

Spring 5-15-2013

Why Ask Why: An Exploration of the Role of Proof in the Mathematics Classroom

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Why Ask Why:
An Exploration of the Role of Proof in the Mathematics Classroom

by
Joanna Rachel Bartlo

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematics Education

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Portland State University
2013

Abstract

Although proof has long been viewed as a cornerstone of mathematical activity, incorporating the mathematical practice of proving into classrooms is not a simple matter. In this dissertation I aim to advance research on proof by addressing this issue. In particular, I explore the role proof might play in promoting the learning of mathematics in the classroom. I do this in a series of three articles (organized as three chapters), which are preceded by an introductory chapter. The introductory chapter situates the remaining chapters in the context of mathematics education research. In the second chapter I explore what the literature on proof tells us about what role proof might play in the promotion of learning in the mathematics classroom. In this chapter I also compare the ways in which proof is purported to promote learning in the mathematics classroom with the roles it is purported to play in the field of research mathematics. In the third chapter I look at empirical data to explore ways engaging in proof and proving might create opportunities for student learning. In particular, my analysis led me to focus on how identifying and reflecting on the key idea of a proof can create opportunities for learning mathematics. The final chapter is an article for a practitioner journal and discusses implications for practice based on the two preceding articles.

Acknowledgements

There are many people I want to thank for their support during this dissertation project. First I would like to thank my advisor, Sean Larsen, for all of the help, guidance, and support he offered me throughout my entire time in the doctoral program as well as with my work on this dissertation. The ideas in this dissertation (and the dissertation itself) benefited greatly from his feedback, and I developed as an educator and a researcher through the support he gave me during the entirety of this program. I would also like to thank Luis Saldanha for the mentoring and opportunities he gave me early in my doctoral career and the thoughtful feedback he gave me during this dissertation process. I feel grateful to have had the opportunity to work with him during the short time he was at Portland State, and grateful that he was willing to continue to work with me from Arizona. I would like to thank John Caughman for taking time to be on so many committees for me over the years, and for always offering thoughtful advice and helpful ideas. I would like to thank Swapna Mukhopadhyay for all she has taught me over the years and for always supporting me through this process. I would like to thank Milan Sherman both for his feedback on this dissertation and for stepping in at the last minute to be on this committee. I would also like to thank Karen Marrongelle for thoughtful feedback on earlier versions of this paper. This dissertation has benefited tremendously from the feedback I received from everyone on my committee.

I also want to thank everyone involved in the JAGUAR project. I cannot thank the teachers enough for opening their classrooms to us and for sharing their thoughts

about justification with us. The graduate student involved also helped make this dissertation possible by spending countless hours gathering data, transcribing classroom video, and organizing the data. So thank you to all of you as well.

Last, but not least I would like to thank my friends and my family. It is nice to know I have so many people to celebrate my accomplishments with and gripe to when I need to. I feel lucky to have such wonderful and supportive friends both within the program and outside of it. I want to thank my parents for all of the support and guidance they gave me through my entire education, especially here at the end when it was not always easy. I want to thank my husband for always being there for me and supporting me in every way imaginable through this grand adventure (and all of our adventures). Finally, I want to thank my sons for reminding me what is important in life and for always cheering me up when I was stressed out or frustrated.

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Chapter 1

Introduction

It is widely accepted among the mathematics education community that proof¹ is an important part of the mathematics classroom. This can be seen in the vast amount of research on the topic (for an overview see Harel & Sowder, 2007) and by recommendations that proof be incorporated into all mathematics courses (e.g. Common Core Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). Most of the large body of research on proof has focused on students' and teachers' conceptions of proof, on learning how to read and write proofs, and on ways to teach proof (Hanna & Barbreau, 2008). In this way, research on proof can be seen to focus on proof as content to learn. Nevertheless, the National Council of Teachers of Mathematics (2000) refers to proof as a process standard rather than a content standard, and they say that "proof offers powerful ways of developing and expressing insights about a wide range of phenomena" (p. 56). In other words, the National Council of Teachers of Mathematics view proof as a process to be incorporated into classrooms to promote student learning, and not just as content that students should learn. If this vision of proof as a process for supporting learning is to be realized, there is a need for research to be conducted from this perspective.

¹ The idea of proof can be seen to encompass both an object (i.e., a proof) and a process (i.e., the activity of proving).

Appealing to Problem Solving Literature

According to Schroeder and Lester (1989), problem solving was the most widely written about topic in mathematics education literature of the 1980s. However, problem solving was not a well-understood area because there was no consensus on what it meant to make problem solving the focus of instruction. Some educators focused on how students solve problems, others focused on how previously learned mathematical content could be used to solve word problems, and others still focused on how problem solving could be used to teach mathematical content. The literature on problem solving refers to these different perspectives as teaching *about* problem solving, teaching *for* problem solving, and teaching *via* problem solving respectively (e.g. Schoenfeld, 1992; Schroeder & Lester, 1989; Stacey, 2005; Weber, 2005).

Schroeder and Lester (1989) argued that the first two perspectives were the focus of much of the research on problem solving, and that these were also the perspectives adopted by most curriculum designers. They advocated that researchers and curriculum designers should direct more attention to the third perspective (i.e., to use problem solving to develop an understanding of mathematics more generally). Over the years since publication of Schroeder and Lester's influential article, a large body of research has focused on problem solving as a way to promote understanding of mathematics. This shift in the research focus is illustrated in a variety of ways, including the vast amount of research focusing on problem solving as a way to promote the understanding of mathematics (e.g. Lesh & Zawojewski, 2007; Stacey, 2005; Weber, 2005) and in the curricula that reflect this perspective (e.g. Fendel, Resek, Alper, & Fraser, 1997; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998; TERC, 1998).

I propose that it is both productive and timely to make a similar shift in research on proof. Just as problem solving—one of the other five process standards recommended by the National Council of Teachers of Mathematics (2000)—was once studied mainly as *content* to learn and is now often studied as a *process* by which one might learn, it is appropriate for researchers to begin studying proof as more than just content to learn, but as a process by which students might learn.

Reevaluating the Role of Proof in the Classroom

Although proof has long been viewed as a cornerstone of mathematical activity, incorporating the mathematical practice of proving into classrooms is not a simple matter. Although proof has been incorporated in mathematics classes at the university level, it has been less evident in K-12 mathematics classrooms. In those classrooms, proof has traditionally been associated only with secondary school mathematics, and mainly with high school courses on Euclidean geometry (e.g. Chazan, 1993; L. Edwards, 1999; Moore, 1994; Wu, 1996). In recent years, however, there has been growing appreciation of the importance of proof in school mathematics at all levels. Despite this emphasis, the role of proof in school mathematics has been unclear (e.g. Simon & Blume, 1996; Steen, 1999). In particular, mathematics educators have been reassessing the role and nature of proof in mathematics education. This reassessment has been influenced by different theories of how students come to know mathematics and by careful consideration of the practices of mathematicians (Yackel & Hanna, 2003). Nevertheless, according to Weber (2010), although there is a fundamental consensus that proof should play a prominent role

in the classroom, it is still not clear what that role should be or how proof could be beneficial to students.

Hanna (2000) argues that one of our key tasks as mathematics educators is to understand the role of proof in teaching. Even if there were a dramatic shift in the nature of research on proof—from viewing proof as a goal of mathematics instruction to viewing proof as a means of achieving other pedagogical goals—it is still not clear what the role of proof would be in the mathematics classroom. This is because there is, as of yet, no consensus as to what those goals might be or how they could be reached.

In an international conference on proof, Balacheff (2002) argued that research on proof was at a standstill because researchers differed in their epistemologies of proof. He argued that if researchers did not begin to articulate their own views, research on proof would remain deadlocked. I believe a similar argument can be made about how researchers view the role of proof as a learning practice. That is, researchers and educators might talk past each other without realizing it if they do not explicate the purpose they see for proof. In order to advance discussions in relation to learning *via* proof, we must begin by understanding how proof can be seen as a learning practice.

Research Questions

The fact that there is far more agreement than ever in the mathematics education community that proof should pervade the school mathematics curriculum offers promise for progress with respect to promoting proof in school mathematics. However, there are still many unanswered questions with regard to precisely what role proof should play in the classroom. Without explicating the different goals that abound in the mathematics

education community, people might be distracted by surface similarities and believe they are working towards a common goal even when they are not. A goal of this dissertation is to uncover the different purposes one might have for incorporating proof in the mathematics classroom. In particular, in this dissertation I explore the different ways proof could serve as a learning practice.

This dissertation will move to advance this goal in two ways. First, by examining the research literature on proof to identify and describe the various roles that proof might play in learning mathematics. Second by examining empirical data to explore how engaging in proof can provide middle grades students with opportunities to learn mathematics. Below I present corresponding research questions for the study:

1. What does the literature on proof tell us about the role proof might play in learning in the mathematics classroom?
2. In what ways might engaging in proof and proving create opportunities for student learning?

To be clear, the overarching research question for this dissertation is what role proof might play in learning mathematics in the classroom. The research questions presented here aim to advance research on proof by beginning to uncover the differences implicit in research on proof, in particular with regards to what role(s) proof might play in the classroom. According to Mamona-Downs and Downs (2005), the main motive for raising an issue is to argue that research should be initiated, extended, or revised in some way. Raising this particular issue responds to Balacheff's (2002) call to explicate

different implicit theories underlying research on proof, so that differences in people's beliefs do not become an obstacle to making progress with respect to research on proof.

In exploring the first research question listed above, I developed a framework that describes the different ways proof is purported to promote the learning of mathematics in the proof literature. The aim is that this emerging framework will serve as a starting point for discussions in the wider community about the myriad of views present in the literature. Exploring the second question will move towards describing the learning opportunities that can be created by proving by displaying specific learning opportunities created by engaging in proof, and showing how proof could lead to those opportunities. These examples will serve as a starting point for discussions about the ways in which students might learn via proving. Both of these studies will help create a platform for further research by raising important issues for discussion, and by coaxing researchers to turn their different views about the role of proof into research questions.

In addition to the goal of helping the proof research community move past deadlock, the research in this dissertation also has more practical aims. For instance, the final chapter in this dissertation is an article for a practitioner journal that shares a few of the most salient findings from the dissertation study with practicing teachers. Sharing these findings with teachers could help teachers broaden their views about what role(s) proof could play in their classrooms. This could help teachers advance their practice with respect to proof. Additionally, to prepare teachers, teacher educators need to understand the complex factors that influence teachers' practice. Moreover, the ideas explored in this dissertation study could help curriculum designers think about what role(s) proof plays in their curricula. This could help curriculum designers better articulate the theories on

which their curricula are developed and better articulate the goals of activities within their curricula, which could be helpful when designing instructor support materials.

I also wish to mention that the focus of this dissertation is on learning *via* proof. It is not my intent to dismiss learning *about* proof and learning *of* proof. There is value in each of those practices in their own right, and those practices are highly relevant to the topic of learning *via* proof. However, there is already a vast amount of research on learning *about* proof and learning *of* proof, and the focus of this dissertation is to engage in discussion about learning *via* proof.

Overview of the Dissertation

This section is designed to orient the reader to the dissertation. This dissertation is organized into a series of three articles with an introductory chapter. Beyond the introductory chapter, the dissertation consists of three chapters- each of which is meant to be a standalone article. Since each chapter is intended to be a standalone article, there may be some repetition within the dissertation. For instance, because each article needs to include the appropriate background for that chapter, there is some overlap in the background covered in each chapter. Additionally, some of the same examples are used for different reasons in multiple chapters.

As mentioned above, the introductory chapter is followed by three chapters, each of which is a research-based article. Chapter two is an article aimed at answering research question one described above and chapter three is a research article aimed at addressing research question two. The final chapter, chapter four, is an article for a

practitioner journal that focuses on implications for classroom practice from the findings presented earlier in the dissertation. These chapters are discussed in more detail below.

In chapter two I draw on the research literature to provide a review of the roles proof is purported to play with respect to the learning of mathematics. In the first section of that chapter, I begin by discussing what I mean by the word proof. Then I discuss the reasons to appeal to the role of proof in mathematics and the consequences of doing so. In the bulk of the paper I discuss an emerging framework about the role proof can play in promoting learning in the mathematics classroom. In doing so, I describe the roles proof plays in mathematics in detail, and I discuss how the research literature suggests those roles can be leveraged to promote the learning of mathematics. I also highlight the differences and similarities between the role of proof in the research mathematics community and in the classroom community. Specifically, I use the literature analysis described in this chapter to reformulate the framework on the role of proof in mathematics described by de Villiers (1990) in a way that is intended to be attentive to the nature of classroom learning as well as to the field of mathematics. Finally I discuss limitations of this emerging framework and ideas for future research it inspires. In sum, this chapter highlights and connects the ways in which proving is purported to promote learning in the mathematics education literature.

In chapter three I use empirical data to illustrate learning opportunities afforded by students' engagement in proof activities. This is based on my analysis of opportunities for learning that occurred in several middle school classrooms. The data used in this study are drawn from the classroom data generated in the first year of the JAGUAR

Project²—a larger project focused on researching justification in the middle grades classroom. I identified moments in the data where students seem to have had an opportunity to learn mathematics as a result of their engaging in proof, and I analyzed how identifying the key ideas of those proofs contributed to the creation of those opportunities for learning. In this chapter I discuss the stories that emerged as a result of that analysis. In particular, I noticed that identifying and reflecting on the key ideas of proofs created opportunities for learning in several different ways. For example, it helped students create new (to the student) mathematics as they generalized the key ideas. It also enabled them to develop new ways of solving problems as they applied the key idea from one proof to a different proof. Additionally, it provided motivation for students to learn about why a statement or fact is true by inspiring them to explore the underpinnings of key ideas. Then I relate these findings to other research on the learning of mathematics. In a sense, this chapter serves as an existence proof that proof can support the creation of opportunities for learning in the mathematics classroom.

In chapter four of the dissertation I reframe a few of the most salient findings from the previous chapters for practicing teachers. The main point I emphasize is that proof can help promote student learning of mathematics in the classroom, and I illustrate this with a classroom episode. In doing so, I describe how a student uses the key idea of a proof that his strategy works to show another student why her strategy does not work. Moreover, by engaging in proving the students are able to revise the incorrect strategy so it does work. In the end, I discuss the role of the teacher and the task in creating these

² JAGUAR is grant funded by the National Science Foundation (NSF) (DRL-0814829). The views expressed in this dissertation are those of the author and do not necessarily reflect the views of the NSF.

opportunities. I also connect the role proof plays in creating these learning opportunities to the role proof plays in the field of mathematics. The goal of sharing these insights is to provide educators with an example of how engaging in proving can create opportunities for student learning in their classrooms.

These chapters together give insight into how proof can promote learning in the mathematics classroom. They summarize existing research on proof as a learning process and extend that research by comparing the arguments discussed in the research literature, by looking at empirical data, and by connecting the findings in both studies to other research on the learning of mathematics. Organizing this research can serve as a starting point for articulating and researching other ways in which proof can promote learning in the classroom, and for articulating other reasons why proof should be incorporated into the classroom. Additionally, articulating the ways in which proof can promote learning helps researchers articulate their perspectives, which can facilitate communication and collaboration among researchers. Moreover it can help teachers, teacher educators, and curriculum developers better understand and articulate their teaching philosophies, which can help with lesson planning, lesson implementation, and, in turn, student learning.

Chapter 2

Literature Review

The Role of Proof in the Mathematics Classroom: How its Role in Mathematics can be Leveraged to Promote Learning in the Mathematics Classroom

Recent standards documents (e.g. Common Core State Standards Initiative, 2009; National Council of Teachers of Mathematics, 2000) have called for the incorporation of proof into all levels of mathematics classes. However, in these calls it is not clear exactly how and why proof should be incorporated in the classroom. For instance, some members of the mathematics education community argue that reading and writing proofs is an important part of mathematics. So, it is a skill students should learn in their mathematics classes (e.g. Selden & Selden, 2003). Others argue that it is not sufficient simply to read and write proofs. Rather, students need to learn about the role proof plays in mathematics (e.g. Hanna, 1995). Both perspectives refer to proof as content to learn, but proof can also be viewed as a process by which students learn (Bartlo, Chapter 1 of this document; Reid, 2011)³. As emphasized by Weber (2010), there is a fundamental consensus that proof should play a prominent role in mathematics classrooms. However, it is not yet clear precisely what that role should be and how proof can be beneficial to students.

³ *Proof and Reasoning* are part of the National Council of Teachers of Mathematics' (2000) process standards, which articulate separate standards for content. However, that could be interpreted as a process students should learn or as process by which students should learn.

This paper aims to shed light on this issue by exploring the potential for proof to serve as a vehicle to promote learning in the mathematics classroom. In particular, I will examine the research literature to identify and describe the various roles that proof might play in learning mathematics. I will relate these findings to the role of proof in research mathematics.

The vast majority of literature on proof focuses on students' and teachers' conceptions of proof, learning how to read and write proofs, and learning about the role of proof in mathematics. These aspects of the literature on proof will not be fully discussed in this paper. Rather, this paper will explore how proof can be seen as a vehicle to promote the learning of mathematics. In fact, there have already been a number of papers synthesizing the research on the teaching and learning *of* proof (e.g. Harel & Sowder, 2007; Stylianou, Blanton, & Knuth, 2009), but the field also needs such an analysis of the research related to teaching and learning *via* proof (Mejia-Ramos & Inglis, 2009).

In reporting the literature on the role proof plays in learning mathematics, I will compare the role proof is purported to play in the classroom with the role proof is purported to play in research mathematics. To do so, first I will discuss what I mean by the word *proof*. Then I will discuss the reasons to appeal to the role of proof in mathematics and the consequences of doing so. Next I will introduce an emerging framework about the role proof can play in promoting learning in the mathematics classroom. Then I will discuss the components of the framework in more detail. In doing so, I will describe the roles proof plays in mathematics in detail, and I will discuss how the research literature on proof suggests those roles can be leveraged to promote the

learning of mathematics. I will also highlight the differences and similarities between the role of proof in the research mathematics community and the classroom community. Finally I will discuss limitations of this framework and ideas for future research it inspires.

Justification, Proof, and Argumentation

For many people the term proof evokes an image of formal mathematical objects. That is, a sequence of deductive steps, based on the axiomatic method, that lead to a desired conclusion. However, this image of proof drastically limits the idea of proof. It is more limiting than the proofs mathematicians actually write (CadwalladerOlsker, 2011), and it is more limiting than what many mathematics educators mean when they use the word proof (Elliot, Leissig, & Kazemi, 2009).

Although some experts distinguish informal arguments using terms such as *justification* and *argumentation* from the more formal idea of *proof*, others argue that the formal format of written proofs and rigor are independent of each other (CadwalladerOlsker, 2011). This suggests that the notion of mathematical proof could be expanded to include what others might consider justification or argumentation (e.g. Elliot et al., 2009).

Formal proofs can be seen as the form that justification and argumentation take in the research mathematics community. In this way, formal mathematical proofs could be seen as arguments subject to the strict rules imposed on final products in the research mathematics community. The format is based on the norms for proofs established in the mathematics community. Since teachers set the norms in their classrooms, classroom

norms can be established in a way that makes proof and argumentation tightly connected (Cobb & Yackel, 1996). These ideas, taken together, suggest that, in the context of mathematics education, justification, argumentation, and proof are not entirely distinct.

Another reason the terms justification, argumentation, and proof often evoke different images is that some people think of proof as a finished product and argumentation as a process. However, mathematical proof, argumentation, and justification can be seen to include a finished product (e.g. a proof) and a process (e.g. the act of proving) (Douek, 1999). So the ideas of proof, argumentation, and justification cannot necessarily be separated by a process/product distinction.

Additionally, in recent years many researchers have tried to broaden the perspective of proof in the classroom from a highly standardized type of argumentation to include a broad range of formal and informal arguments. Consequently, there are many ways in which the ideas related to proof, argumentation, and justification can be seen as interconnected. In a sense, they could be seen as parts of a continuum rather than as distinct notions (Hanna & deVilliers 2008).

As the meaning and scope of proof are not fixed by a technical definition, it can be difficult to determine what a given author means by the terms proof, justification, and argumentation. This creates challenges in drawing firm conclusions about what the literature says about the role of proof in the classroom. Therefore, in this literature review I will draw on the literature related to all three concepts: proof, justification, and argumentation.

For the purposes of this paper, I will define proof as *removing doubts about a claim* (Davis, 1986). This generally involves articulating evidence and warrants in

support of a claim to convince oneself or others (Toulmin, 1958). This process involves more than an explanation of how a person arrived at an answer; it also includes a reason that or why the answer is true. This reason could be supported in a variety of ways, including informal and deductive arguments. For instance, if a person describes that they found an answer by using the standard algorithm for addition, they would be explaining how they found their answer but not proving that or why it was the correct answer. However, if the person described why that procedure worked, they would be offering a proof for their answer.

While authors do not always clearly state what they mean by the terms they use in their publications, I will try to examine the generalizability of the ideas expressed in each publication to the different conceptions of proof. To be faithful to the research, when referring to specific studies I will use the language from that study. For example, I will use the word justification if the author uses the word justification, and the word proof if the author uses the word proof. In writing about my own ideas, I will use the three words interchangeably, and will refer to the rigorous version of proof that is commonplace in the research mathematics community as *formal* proof to distinguish this idea from other uses of the word proof. When I mean to exclude the rigorous or formal version of proof commonplace in the mathematics community, I will differentiate the more informal incarnations by using the word *informal* as a modifier.

In a similar vein, my intention in this article is to discuss what role proof can play in the promotion of learning in any level of mathematics instruction. To this end I discuss studies that were conducted at various grade levels. In my discussions of the studies I state at what grade level they were conducted, and where appropriate, I

hypothesize why the studies may or may not generalize to other grade levels. However, it is possible these differences are more attributable to classroom norms than they are to grade level. For instance, since teachers can establish norms in a ways that make the practices in their classrooms more or less tightly connected to formal proof (Cobb & Yackel, 1996), it is possible a theory could generalize among classrooms with similar norms at different grade levels rather than among classrooms with different norms at the same grade level. That said, since the goal of this article is to explore the ways proof is purported to promote learning of mathematics in the research literature, the focus of this paper is to discuss the myriad of ways proof can be seen to promote the learning of mathematics rather than to articulate conditions under which each role is most effective. Those questions are left for future studies.

Method

The goal of this paper is to identify different ways in which proof is purported to promote learning in the mathematics classroom. For the purposes of this literature review, I searched for articles that discuss the notion of mathematical proof, justification, or argumentation in relation to learning in the mathematics classroom. Following Mejia-Ramos and Inglis (2009) , I conducted a search using only the Education Research Information Center (ERIC) database in order to avoid sampling bias. I conducted ten Boolean searches using combinations of the keywords *learning* and *understanding* with the keywords *proof*, *proving*, *justification*, *justify*, and *argumentation*. This search (conducted April 2012) identified nearly 3000 articles, the vast majority of which were irrelevant to this study as they did not address the connection between proving and

learning (e.g. articles involving justifications for learning about the classics, articles discussing proof that learning occurred in various contexts, articles discussing how to help students learn to prove, and articles about automobiles)⁴. After reducing the list to only the articles that actually addressed the topic at hand⁵, my final sample contained 29 articles.

I then searched the reference lists of each of these articles to see if any relevant literature was missed. Most of the literature referenced in these articles was already found in the ERIC search or was not directly relevant to the topic of this study (e.g. these articles focused on policy discussions, different teaching practices, textbooks used in the study, constructs underlying the research, and learning about proof). However, there were two relevant articles and two relevant book chapters referenced in the articles found from the ERIC search. Consequently, the literature used in the literature review includes the 31 articles and 2 book chapters found from this two-phased search.

After finding, reading, and summarizing the literature in the sample, I classified each article with respect to the role proof was purported to play in learning mathematics. After identifying those themes, I compared the emerging themes about the role proof plays in learning mathematics to the roles proof is purported to play in the field of mathematics itself (de Villiers, 1990), which led to the framework described in the following section.

⁴ Including the word mathematics in the search would exclude many of these irrelevant articles, but it also excluded many relevant articles since the word mathematics is usually not included as a keyword in mathematics education research articles.

⁵ Most of the articles were eliminated immediately, as it was obvious from the title they were irrelevant. A few were eliminated based on their abstract or on perusal of the articles themselves.

The majority of the articles (about 2/3) used in the literature review are qualitative, and the rest are theoretical; none are quantitative. Of the qualitative studies, about half involve elementary or middle grade students, and half involve secondary or post-secondary students (one uses examples from both levels). The theoretical studies often refer to advanced examples, but do not directly address any grade level in particular.

The Role of Proof in Mathematics

De Villiers' (1990) seminal work has provided one of the most influential frameworks with respect to the role of proof in mathematics (Yopp, 2011). The five roles of proof de Villiers describes are: (i) proof as a means of *verification*, (ii) proof as a means of *explanation*, (iii) proof as a means of *systematization*, (iv) proof as a means of *discovery* and (v) proof as a means of *communication*. Although many other papers have been written about the role of proof in mathematics (e.g. Hanna, 2000; Jaffe, 1997; Thurston, 1995), most of the functions mentioned in those articles could be characterized by one or more of the roles described by de Villiers (Yopp, 2011). For this reason, the framework described by de Villiers (1990) is often used to discuss the role of proof in the mathematics classroom as well as in the field of mathematics (e.g. Hanna, 2000; Herbst, Miyakawa, & Chazan, 2012; Knuth, 2002b; Staples, Bartlo, & Thanheiser, 2012). Although this framework is often used to describe the role of proof in the mathematics classroom, it does not provide a full description of the role of proof in mathematics classrooms (Staples et al., 2012; Yopp, 2011). In the following section, I discuss this issue in more detail.

Appealing to Mathematics

Some researchers argue that the role of proof in the mathematics classroom should reflect its role in the field of mathematics (e.g. Chazan & Lueke, 2009; Hanna, 1995; Knuth, 2002c; Stylianides & Stylianides, 2006). Additionally, some would argue that proof is one of the main ways mathematicians come to know mathematics (e.g. Chazan & Lueke, 2009; Knuth, 2002c; Thurston, 1995), and that considering how the discipline comes to learn mathematics could provide insight into how students could learn mathematics in the classroom (Wilkerson-Jerde & Wilensky, 2011). Consequently, an understanding of the role proof plays in the field of mathematics could inform educators' perspectives on how proof can promote learning in the mathematics classroom. For this reason, de Villiers' (1990) framework is often used to frame discussions of the role of proof in mathematics classrooms as well as in the field of mathematics.

However, there is debate among mathematics educators as to whether all five purposes described by de Villiers (1990) are relevant in the mathematics classroom. In fact, de Villiers (1990) himself asks which of these functions can be utilized in a mathematics classroom to make proof a more meaningful activity. Additionally, Wood (2001) argues that children's thinking does not always follow the structure of mathematics, and that teaching involves knowing both where children's thinking and the field of mathematics converge and where they diverge. Therefore, knowing how proof promotes learning in the field of mathematics may not accurately describe how proof can promote learning in the mathematics classroom. In short, de Villiers' framework could

be a useful first step in describing the role proof could play in the classroom, but it may not attend to some important nuances of classroom practice.

Moreover, the role of proof in the field of mathematics does not provide a sufficient framework to understand the role proof might play in the classroom is because there are functions in the classroom that are not reflected in the use of proof in the context of the research mathematics community. For instance, in a study of 4th – 9th grade teachers, Staples and Truxaw (2009) identified assessment of student understanding as an important purpose of proof in the classroom context. Assessment of understanding is not one of the five roles proof is purported to play in the mathematics community, but it could be argued that this assessment purpose is related to proof's role of communication in research mathematics. However, simply describing this use of proof as communication loses much of the important nuance of how proof is used by these teachers.

In short, because teachers' goals are different from those of research mathematicians, proof plays additional roles in the classroom beyond those captured by the five roles described in de Villiers' (1990) framework. In what follows, I propose a modified version of de Villiers' (1990) framework that is intended to more accurately reflect the role proof might play in promoting the learning of mathematics in the classroom. In doing so, I explore the roles proof is purported to play in promoting learning in the mathematics classroom and look at the commonalities and differences between the role proof plays in research mathematics and in the mathematics classroom.

The Emerging Framework

Most research addressing the role of proof in the mathematics classroom uses the roles proof plays in mathematics (e.g. de Villiers, 1990; Hanna, 2000; Herbst et al., 2012; Knuth, 2002c). However, research using this framework has the potential to determine whether proof plays the same role in the mathematics classroom as it does in mathematics, but does not elucidate distinctive roles proof might play in the classroom (Staples et al., 2012; Yopp, 2011). Since my goal is to explore the roles that proof could play in learning mathematics, I recast de Villiers' (1990) framework to describe the role proof can play in learning mathematics. In doing so, I show that proof is used for similar purposes in the mathematics classroom as in the mathematics community, but that only looking at those roles as they function in research mathematics overlooks many of the intricacies of the role of proof in promoting learning in the mathematics classroom. *Figure 1*, below, shows the emerging framework resulting from the literature search and analysis described above. The framework will be discussed in more detail in the following sections.

Role of Proof		Elaboration of the Role of Proof as a Learning Practice
Verification	Conviction	Proof can remove doubt, or convince a person of a claim. As students validate their own ideas, this role of proof can be leveraged to help students become autonomous learners.

	Confirmation	Proof can confirm that a claim is true or that a procedure is appropriate for a given situation. This can be leveraged to help develop a classroom community's shared knowledge.
Explanation	Insight	Proof can provide insight into why a mathematical phenomenon is true and how it works.
	Consequences	Proof can show how one result is the consequence of another result. This can help students learn mathematics because applying a concept in a proof can help students see the consequences of the concept.
Systemization	Inconsistencies	Proof helps uncover inconsistencies, circular arguments, or implicit assumptions. In this way, proof can help students learn mathematics as they encounter and respond to counter-examples and counter-arguments.
	Connections	Proof can unify and simplify theories by connecting unrelated statements. This can help students build a connected network of ideas.
	Global Perspective	Proof can provide a global perspective on a topic by exposing its underlying structure.
	Application	Proof can aid in the discovery of applications of ideas and theories. In the classroom this may mean learning how to generalize solutions from one problem to other problems.

	Alternative Systems	Proof can lead to alternative deductive systems that are more elegant or powerful than existing ones. In the classroom this can mean learning new methods for solving problems and new ways of thinking about problems.
Discovery	Exploration	Proof can lead to new results by enabling exploration of the consequences of assumptions. This can help students learn new ideas by enabling exploration of their existing knowledge.
	Analysis	Proof can illuminate the key idea of an argument, which can be generalized to create new results. Reflecting on the key idea can also promote students to learn about the key idea itself.
Communication	Form of Discourse	Proof is the way mathematical results and reasoning are communicated in the field of mathematics. This can help students learn because it puts students' ideas on display for the teacher and for other students.
	Forum for Debate	Proof is how the mathematics community determines if claims are true and arguments are valid. This can help students refine and revise their ideas.

Figure 1. Elaboration of the role of proof in research mathematics community and the role of proof in promoting learning of mathematics in the classroom.

The five functions listed on the left hand side of the framework are the five roles of proof de Villiers (1990) describes in his paper on the role of proof in mathematics.

Although these five functions are the most cited part of his paper, the paper also describes subcategories for each of the functions. Although de Villiers did not name the

subcategories, for ease of communication I used keywords to name each of these subcategories. The middle column of the framework displays the subcategories de Villiers (1990) describes⁶.

The right hand column of the framework describes the role proof is purported to play in the field of mathematics, since proof can, potentially, play a similar role in the promotion of learning mathematics in the classroom. This column also displays ways in which the roles proof plays in the field of mathematics can be leveraged to reach other educational goals in the classroom. These purposes are drawn from research on the role proof is purported to play in the learning of mathematics that is discussed in the research literature on proof and on the learning of mathematics.

It is worth noting that this chart shows several ways proving relates to learning in the mathematics classroom. First, proof can promote the learning of mathematics in the mathematics classroom in precisely the same way it does in the field of mathematics. Second, the descriptions of the role proof plays in the field of mathematics could be seen to describe activities students can engage in. From a social perspective on learning, these activities could be seen to constitute learning in itself. Learning can be thought of both as the activity of acquiring knowledge and the knowledge that is acquired (Sfard, 1998). Researchers who view learning as an activity often consider the act of justifying or proving to be learning itself. For instance, some researchers argue that participating in practices such as defining, symbolizing, algorithmatizing, and justifying (and changes in the ways in which students engage in such practices) constitutes learning (e.g. Rasmussen

⁶ In some cases de Villiers (1990) explicitly carved out the subcategories described in the framework. In other cases, I analyzed his descriptions of the roles to identify subcategories.

& Marrongelle, 2006; Rasmussen, Zandieh, King, & Teppo, 2005). From this perspective, the right hand column of the framework could be seen to describe activities that constitute learning. The third way the right hand column of the chart relates proof and learning is that it describes the unique ways educators use proof in the mathematics classroom. In particular, based on the research literature that connects proof and learning, the right hand column of the proof describes how teachers can leverage the roles proof plays in mathematics to promote learning in the classroom. That is, it is because proof plays the given roles in the field of mathematics that proof can be seen to play a corresponding role in the mathematics classroom. Consequently, each row of the framework describes several ways in which proof can be seen to promote learning in the mathematics classroom: the same way it promotes learning in the field of mathematics, an activity that constitutes learning, and a way the role proof plays in mathematics can be leveraged to promote the learning. Of mathematics in the classroom.

Additionally, although the framework may suggest otherwise, there is overlap between the rows. It can be hard to articulate the differences between the roles of proof in such a way that the differences between them are clear (Yopp, 2011). Moreover, when this framework is extended to show the ways in which proof's functions are leveraged to reach educational outcomes, even more overlap arises. That is, two different roles of proof can be leveraged to create similar learning opportunities. Consequently, the functions described in this table are not necessarily entirely distinct. Nevertheless, this organization is still useful because it highlights important aspects of the role proof can play in the classroom as well as the differences between them. The similarities and

differences between related functions will be discussed in more detail throughout this paper.

One could speculate that proof can play additional roles to promote learning in the mathematics classroom. However, the point of this framework is not to create a comprehensive list of the role proof can play in the promotion of learning mathematics in the classroom, but rather to describe what the research literature says about the role proof can play in the promotion of learning in the mathematics classroom. In this way, this framework can be seen as a starting point towards describing the role of proof in promoting learning in the mathematics classroom.

Additionally, proof is not the only way to reach some of these learning goals. The goal of this framework is not to describe ways proof uniquely promotes learning, but rather to discuss the ways in which proof can be seen to promote learning (uniquely or otherwise). Throughout the paper I hypothesize reasons why proof might be a particularly good vehicle for creating certain learning opportunities, and discuss when other processes might be just as effective.

In the following sections, I describe the components of the framework in more detail, including descriptions of the literature from which this framework is drawn. In discussing the framework and the literature, I will discuss the similarities and differences between the roles proof is purported to play in the field of mathematics and the roles it is purported to play in the mathematics classroom⁷. In order to discuss both the framework

⁷ The goal of this paper is not to analyze the role of proof in the field of mathematics, but rather to highlight the roles it can play in the math classroom. Therefore, it is possible that at times I may oversimplify some of the nuances of the role of proof in the field of mathematics.

and the literature in more detail, I use de Villiers' (1990) framework to organize the discussion of the literature.

The Role of Proof in Mathematics Education

In this section I will describe the aspects of the framework shown above in more detail. The framework is shaped by the roles proof is purported to play in the field of mathematics, but it aims to show how those roles can be leveraged to promote the learning of mathematics in the classroom. That is, by looking at the roles proof is purported to play in promoting learning in the mathematics classroom, I hope to recast de Villiers' (1990) framework in a way that is attentive to educational goals as well as to proof's role in mathematics.

Verification

The first role proof plays in the field of mathematics, as described by de Villiers (1990), "concern[s] the truth of a statement" (p. 18). De Villiers calls this role of proof verification. Proof serves as a means of verification in the field of mathematics in two ways: it can remove doubt about a claim and it can confirm that a claim is true. I call the two subcategories proof serves as a means of verification *conviction* and *confirmation*. Proof can play the same role in the mathematics classroom. That is, proof can promote the learning of mathematics by convincing students of the veracity of theories or claims and by helping students confirm theories or claims just as it does for mathematicians. Moreover, the ideas of confirmation and conviction can be leveraged in the classroom to help serve other educational purposes. For instance, the fact that students can gain

conviction through engaging in proof can help students become autonomous learners in the classroom. Additionally, the fact that students can confirm (or refute) claims helps groups of students discuss mathematical ideas. These ideas are discussed in more detail in the following sections.

Conviction. Many mathematics educators argue that students are convinced of many statements without the use of proof (Harel & Sowder, 1998). However, that does not mean that students cannot be convinced of the veracity of statements by proving. If a student is comfortable with the deductive style of formal mathematical proofs, a student may be able to gain conviction about a mathematical claim by using such a process. However, many students may not be convinced of the veracity of a statement by engaging in such a process. Nevertheless, those students may be able to verify a conjecture or claim by engaging in a more informal proving process.

If students can gain conviction about claims for themselves through engaging in proving, students can develop intellectual autonomy because they do not need to rely on external factors such as teachers or textbooks. It is possible that a student could gain conviction by making sense of a proof that a teacher, a textbook, or another student presents them- not because they trust an authority but because they make sense of the proof presented. This type of sense making is an important part of the proving process, and often involves recreating the proof for oneself (Weber, 2008)⁸. Intellectual autonomy is an important characteristic for students to have if they are to be involved in

⁸ Throughout this paper when I talk about the act of proving, I include both the act of creating proofs and the act of making sense of (or re-creating) someone else's proof.

creating and understanding mathematics rather than memorizing procedures. For this to happen, students need to rely on logic, mathematical reasoning, and mathematical evidence (National Council of Teachers of Mathematics, 1991). In this way, proof can promote learning in the mathematics classroom because it helps students become convinced about mathematical ideas.

In a study of pre-service teachers, Simon and Blume (1996) discuss the relationship between “the investigation of mathematical validity and the development of mathematical understanding” (p. 3). They draw their data from a three-year project studying the mathematical and pedagogical development of 26 pre-service elementary teachers. The study involved a mathematics course that employed a teaching experiment methodology and one of the focuses of the course was the idea that mathematical knowledge is constructed and validated by the community. The researchers observed that at first the participants relied on previously learned information and appealed to the teacher for verification of ideas. Over the course of the teaching experiment, the subjects became self-reliant and began to validate theories on their own. In this way, the researchers saw that, through engaging in justification, the participants became autonomous learners as they were able to convince themselves of the veracity of mathematical ideas. In this way, the fact that proof serves as a means of conviction helps students become autonomous learners. So teachers can leverage the fact that proof serves as a means of conviction to promote the learning of mathematics by helping students become autonomous learners.

Yackel and Cobb (1996) also explored the relationship between argumentation, and the development of intellectual autonomy. Their research studied how elementary

students became intellectually autonomous in mathematics. They, too, saw that at first the research participants relied on status or authority to determine the veracity of statements, but later they validated their own ideas. Yackel and Cobb explain that the ability to validate one's own ideas contributes to the development of intellectual autonomy. For students to become autonomous learners, they need to take over some of the teacher's traditional responsibilities, including determining what counts as acceptable solutions. Therefore, justification and argumentation play a large role in developing autonomy in students by providing conviction about mathematical ideas.

Another important part of becoming an autonomous learner is metacognition. In order for a student to be autonomous in the classroom, she needs to keep track of what she is doing and why she is doing it. By drawing on evidence from cognitive psychology about how students solve problems, Lucast (2003) argues that proof does precisely that. In particular, she argues that proof promotes metacognition in students because it causes students to ask why they are doing what they are doing. Additionally, since proof shows how one fact follows from another, it helps students think about their thinking by helping them identify the successful steps they have taken in solving a problem.

By engaging in proving, students can gain conviction about a theory. This conviction is intrinsically valuable in its own right, but the process of engaging in proof can also promote learning in the mathematics classroom. There are subtle differences between the way proving provides conviction in the field of mathematics and in the mathematics classroom. At times mathematicians appeal to authority and trust the proofs of others, but mathematicians have the task of expanding the field's body of knowledge. Therefore, mathematicians cannot rely upon textbooks or other experts to validate their

truly novel ideas. Rather, they can only be convinced of the soundness of these ideas by engaging in mathematical practices such as proving. On the other hand, mathematics students are operating within an already established field and students are aware that, often, their answers can be validated by a textbook or teacher. Consequently, students do not see proof as their only recourse for conviction. When students validate their own ideas without appealing to authority, they become autonomous learners. Even if the knowledge is already known to others, the goal is for the student to gain conviction following the same creative path a mathematician would have taken when discovering that knowledge for the first time.

Confirmation. In addition to providing conviction about mathematical ideas at a personal level, proof can also be used to verify ideas in a more public forum. In the mathematics classroom, this can mean building a community's shared knowledge. This could be seen as providing confirmation of an idea at a communal level, or it can be seen as leveraging the idea of confirmation to help create a shared knowledge base for a classroom community.

Stephan and Rasmussen (2002) observed this in a study involving a college level differential equations class. They observed that students introduced ideas in discussions that were not initially accepted at the classroom level. As students argued about those ideas, the ideas were discarded, revised, or accepted. When ideas were accepted at the class level, Stephan and Rasmussen said that *math practices* were established. That meant those ideas were taken-as-shared within the classroom community and could be referred to without further justification. In other words, argumentation allowed the class

as a whole to reach closure with respect to ideas introduced by members of the classroom community. When proving is being used in this way, the purpose of proving is to help provide confirmation of a claim.

By confirming claims, the process of proving can promote learning of mathematics in multiple ways. Because proving provides confirmation about the veracity of statements, it can help students settle debates about mathematical ideas under consideration in classroom discussions. Moreover, proving helps to create a shared body of knowledge within a classroom community.

Summary. Proving can help students verify that statements are true, and, in turn, can help students learn mathematical content. In the literature discussed above, proof was used as a means of verification – thereby, helping individuals and classroom communities gain conviction about and confirmation of mathematical ideas. This is much like what happens in the mathematics community as well. An individual mathematician comes to know a new idea by verifying it for himself via proof (one's own or another person's), and the mathematics community comes to accept a new idea by debating it via proof at the community level. Although both students and mathematicians use proof to establish facts at a personal and a communal level, students are operating in a terrain in which they are not experts. Mathematicians already have confidence in the processes and facts they are using, whereas students are aware there are external resources for verifying their results and have to gain confidence in their ability to verify results. Consequently, in the mathematician community the emphasis is on the a

posteriori verification while in the classroom community the emphasis is on the fact that there is a process by which that verification can happen as well as on verification itself.

Many researchers argue that proof is mainly used for verification purposes in the classroom, but that it should be used for more than that (Hanna, 1990). While that might be true, the verification purpose of proof clearly has a powerful potential to promote learning in the mathematics classroom. This is especially true if, as is described above, verification is considered from a slightly different vantage point than it is in the field of mathematics.

That said, for proof to play a role related to verification in the classroom, students must be convinced by proofs. As I mentioned in the beginning of this section, studies have shown that students are often convinced by examples and not by proofs. So it cannot be taken for granted that engaging in proving will lead these results. Rather, it is important that students are enculturated into the process of proving. In this way, students need to learn to prove and about proof while they are learning via proving.

Explanation

Explanation is the second function de Villiers (1990) describes in his article. Although the litmus test of whether or not an argument is a proof is whether or not it verifies a claim, mathematicians often turn to proofs when they want to know why something is true rather than just that it is true. Proof serves as a means of explanation because it provides insight into why something is true and an understanding of how an idea is a consequence of a familiar result. I will call these two subcategories *insight* and *consequences*. Proof can serve as a means of explanation in the classroom in the same

way it does in the field of mathematics. In fact, many mathematics educators argue this is the most important role proof can play in the classroom. These ideas are discussed in more detail in the following section.

Insight. Many mathematics educators argue that explanation is the most important purpose proof can serve in the classroom (e.g. Hanna, 2000; Hersh, 1993; Knuth, 2002a). They argue that it has been widely shown in the literature that many students are convinced by empirical evidence (for an overview see Harel & Sowder, 2007), but such evidence does not help students gain an understanding of the mathematics in the problem. Consequently, many mathematics educators value proof as a classroom practice because it can help a student understand why something is true.

Although many agree that the explanatory role of proof should be harnessed in the classroom (e.g. Hanna, 1990; Knuth, 2002a), there is no agreement in the literature as to what makes a proof explanatory (Raman, 2003). Some authors offer examples of proofs that they believe explain or of proofs that they believe only prove (e.g. Hanna, 1990; Knuth, 2002a). These often involve showing proofs that involve pictures and stating that the visual proofs explain what they are proving, and showing proofs by induction or involving complicated algebra and stating that those proofs only show that a statement is true without offering any explanation. A few authors also offer definitions of what makes a proof explanatory. These are discussed in more detail below.

Hanna (1990) says that a proof explains when it “reveals and makes use of the mathematical ideas which motivates it” (p. 10). By that she means it shows the property on which the results of the proof depend. She says the types of proofs she describe help

students learn *why* something is true rather than just *that* it is true. One criticism I have of Hanna's definition is that a proof might be explanatory to one person and not to another. This is because different people attribute different meaning to different mathematical objects and because people's personal understandings of the concepts in the proof would impact their personal understanding of the proof. This could have several implications for an individual's understanding of a proof, one of which might be that one person may find that a proof uses the mathematical property that motivates it while another person might not.

Weber (2010) proposed a different definition: that explanatory proofs allow the reader to translate a formal argument to a less formal one. This definition seems more personal, as it could be read to apply to whether or not a proof is explanatory to a given person. Although this definition offers a potentially useful criterion for evaluating proofs, it may just be describing a subset of explanatory proofs. For instance, students may find a formal proof explanatory without having to translate it into an informal system. Also, this definition does not offer criteria for evaluating informal proofs.

Although it is clear that the research community values explanatory proofs because of their role in promoting insight about a topic, the concept of explanatory proofs is yet to be well-defined. More work needs to be done to explore which proofs can be seen to promote insight into ideas. Moreover, much of the work addressing explanatory proofs seems to focus on formal proofs, and neglects informal ones. I do not believe that only formal proofs can promote insight into ideas, nor do I believe that all informal proofs can accomplish this objective. Therefore, the community should not only seek to

clarify what makes a formal proof explanatory, but what makes informal ones explanatory as well.

Additionally, Reid (1995) argues that using proof as a means of promoting insight has some limitations. In particular, he argues that deductive proving is only one form that explanation takes in the field of mathematics. He argues that another form it takes is in explanation by analogy. Consequently, his complaint about the role of proof in promoting insight might be mitigated by expanding the notion of proof to include more than just formal proof.

Most of the literature that addresses the issue of proof providing insight offers theoretical definitions about which proofs do that, but some studies do show that engaging in proving can help students gain insight. In particular, in Simon and Blume's (1996) paper about prospective elementary school teachers the authors argued that when students engage in justification, they have rich opportunities for understanding mathematics that result from involvement in the creation and validation of ideas. In their study, they saw that when the students were pushed to justify their answers they had to look more deeply at the procedures they were using to evaluate the appropriateness of their procedures. As a result the students had to explore why their methods worked, which provided the students insight about the procedures they were using and the concepts they were exploring. So there is empirical evidence that justification can promote learning in the mathematics classroom because it can help students gain insight into the topics they were studying.

Additionally, the focus of discussions about proofs that provide insight seems to be on the explanatory nature of proofs, and not the value of the proof itself. That is, these

authors are asking the question of what role proof can play in the classroom, and the answer is they can offer explanations. However, if the main reason for including proof in the classroom is because it offers explanation, it begs the question of if explanations themselves would be adequate or if explanatory proofs offer something more than other explanations. If one is trying to make the argument that proofs that provide insight play an important role in the learning of mathematics, there needs to be an emphasis on the importance of the proof part of explanatory proofs as well as explanatory part.

Consequences. Another way proofs can provide explanations is that proofs can show how one result is a consequence of another. The authors that explicitly address how proof promotes the learning of mathematics in this manner mainly argue that by connecting unrelated statements proof can help students build a connected network of ideas. They also argue that by enabling exploration of the consequences of assumptions, proof can help students expand their existing knowledge. This literature is discussed in more detail later in both the *Connections* section (*Systemization*) and the *Exploration* section (*Discovery*).

Students can learn about consequences of mathematical theorems and definitions in a more nuanced way as well. Seeing what the consequences of a concept are when it is used in a proof can help students learn about the concept being used in the proof. For example, Hanna and Jahnke (Hanna, 2000; Hanna & Jahnke, 1993) theorize that proving can create opportunities for learning in such a manner. By looking at specific advanced mathematics problems, they present ways they believe that students can explore mathematical definitions and consequences of assumptions while engaging in proving

activities. They elaborate on this idea by saying that students learn about theorems and definitions by applying them in proofs. The authors argue that since theorems and definitions often appear in terms of formal properties without intuitive meaning, the meanings and implications of the theorems and definitions become clear when they are used in different proofs because the application of the theorems and definitions to the new problem puts a new perspective on the definitions and theorems.

Chin and Tall (2002) also explain that for a theorem to be useful to a student, the student needs to use the theorem in a proof because it allows the student to see the consequences of the theorem. Their argument is based on thinking of proof as a *procept* (Gray & Tall, 1994). A *procept* involves thinking of a mathematical entity as a process that produces a mathematical object, as a mathematical concept, and as a symbol that represents the process or concept. As Chin and Tall explain, a theorem can be thought of as a symbol that represents the concept it describes, and the theorem becomes a concept when it is used in a proof of another theorem. The proof involving the theorem is the process, as it shows the theorem “in action”.

In their research with undergraduate students, Chin and Tall (2002) saw that for a theorem to be useful to students, the students have to have a robust *concept image* (Tall, 1981). Part of that image involves seeing the theorem as a process, as a concept, and as a symbol. That is, students need to be able to unpack the notion of the theorem and to be flexible enough with it to apply it to future problems in different ways. Thus, thinking of a theorem as a procept helps students have usable knowledge of it. Therefore, proving a theorem helps students develop a proceptual view of the theorem, and, in turn, to learn about the theorem or concept in a way that allows them to use it in the future. In this

way, proving can be seen to help students reflect on a mathematical concept they have already studied, and to help them learn more about that concept.

All of the research discussed in this section is based on the idea of formal mathematical proof, but there is no reason these ideas would not extend to less formal proofs as well. In less formal incarnations of proving, students still apply and use concepts, theorems, and definitions. This implies that such proving activities could still create opportunities for students to reflect on and learn about their existing knowledge. For instance, when a child proves that the sum of two odd numbers is even she will likely rely on the definitions of odd and even numbers. In applying the definitions to her proof (even an informal working definition), she would have the same opportunities to reflect on the definitions as students applying more advanced and formal definitions would.

In both of these articles, the key seems to be in applying the concept, and not in applying it to a proof per se. This leaves the question of if it is the application of the concepts that yields these benefits, or if it is the application of concepts *to proofs* in particular that yields the learning described in these articles. It is possible there is a category of applications (of which proof is one) that serve this purpose, but it is also possible that in some way proof plays a special role in learning in this way. For instance, it could be the fact that proving can promote metacognition (Lucast, 2003) that motivates students to reflect upon the concept in use to see how it leads to the solution, and in turn to learn about the concept. Nevertheless, this is a theory that should be explored empirically.

Summary. Hersh (1993) argues that the primary function of proof is to *convince* in the field of mathematics, and to *explain* in the classroom, and many other researchers echo this emphasis. However, most of the literature that addresses the explanatory role of proof in the classroom involve theoretical definitions of explanatory proofs, examples of proofs that the authors consider to be explanatory or not, or theoretical ideas about how proofs can help students understand mathematical ideas. More research needs to be conducted on this topic, to explore how proof can help promote insight and can help students see the consequences of concepts used in proofs.

Systemization

Although some people can gain conviction about or insight into an idea through empirical explorations or intuition, proof is an essential tool for systemization (de Villiers, 1990). When considering systemization, the point is not to check for certainty, but to organize individual ideas into a coherent system and to expose the underlying logical relationships between statements. De Villiers (1990) argues that proof is the only tool that can do this. By referencing an earlier article (de Villiers, 1986), he elaborates five ways in which it does so. First, proof can identify inconsistencies, circular arguments, or implicit assumptions. Second, proof can connect unrelated ideas. Third, proof can expose the underlying structure of a topic. Fourth, proof can aid in the application of mathematical ideas, to topics both inside and outside of mathematics. Fifth, proof can lead to “alternate deductive systems which provide new perspectives and/or are more economical, elegant and powerful than existing ones” (p. 21). I named

these subcategories *inconsistencies*, *connections*, *global perspective*, *application*, and *alternative systems* respectively.

Inconsistencies. Proof can help uncover inconsistencies. In the classroom, this can happen as students encounter counter-examples and counter-arguments. Research has shown that students can learn mathematics as they respond to counter-examples and counter-arguments (Balacheff, 1991; Larsen & Zandieh, 2008; Leita, 2000). Below I will elaborate how some studies show that proof can help students learn mathematics as they deal with inconsistencies in their own personal mathematical systems.

Some research shows how proving encourages students to revise misconceptions they may hold about mathematical ideas. This research largely draws on the *constructivist* theory of learning (Piaget, 1971; von Glasersfeld, 1995). Constructivism suggests that people construct their own understandings of the world by reflecting on their own experiences. Individuals generate their own mental models of the world around them, which they use to make sense of their experiences. Learning, then, occurs when people adjust their models in response to new experiences. In particular, as new experiences are encountered, they are incorporated into an already existing framework. When a situation has an unexpected result, a *perturbation* occurs and a person experiences *disequilibrium*. The constructivist learning theory is based on individuals eliminating *perturbations*. Learning is said to occur when an individual makes an *accommodation* to eliminate a *perturbation*. Studies conducted within this constructivist framework focus on the role of *perturbations* and *disequilibrium* in learning to explain

how proving encourages students to revise their existing knowledge. In other words, research in this vein addresses how students deal with inconsistencies they experience.

Balacheff (1991) appeals to the constructivist theory of learning when he states that students actively construct their own knowledge by overcoming disequilibrium. Consequently, counterexamples play an important role in the learning of mathematics. When students have false claims in their proofs, they may be able to uncover counterexamples to their existing theories. This can cause students to experience *perturbations*, leading students to reorganize their existing knowledge. This argument is drawn from data from pairs of 13-14 year old students designing a way to calculate the number of diagonals of a polygon once the number of vertices is known. Once the students believed they had found a solution, the observer offered them a counterexample to their solution, which elicited a variety of responses from the students. For example, some students dismissed the counter-examples, while others restricted the domain of the problem to eliminate the counter-examples. However, some students recognized that they were unsure of the precise definitions of *polygon* and *diagonal*. Those students tried to revise their definitions of those terms in light of the counter-examples with which they were being presented. This allowed them to dismiss the counter-examples which were not actually counter examples and helped them to refine their conjectures. In this way, proving promoted learning by helping students encounter, and ultimately overcome inconsistencies in their own mental models.

Research on this topic is not limited to the field of mathematics. When discussing everyday argumentation, Leitaó (2000) argues that engaging in argumentation creates *perturbations* for people to overcome and that giving people opportunities to rectify their

disequilibria helps them construct new knowledge. She gives examples of everyday arguments, and shows that when people have counter-arguments presented against their arguments, they often revise their original stance after careful consideration of the counter-argument. She attributes the changes in views to an *accommodation*, and therefore calls it learning. Forman (2000) argues that it might not be appropriate to extend this research to the field of mathematics education, since different communities define learning, knowledge, and argument differently. However, Leitao's findings are similar to Balacheff's claims, but differ in that she is talking about a larger group of *perturbations*. That is, she opens the field to counter-arguments and not just counter-examples. It seems that counter-arguments could have the same impact on student thinking as counter-examples, and could play a similar role in helping students uncover inconsistencies in their thinking. Therefore, Leitao's findings have the potential to offer insight into a way in which proof may foster learning in the mathematics classroom. Further research should be done to explore the applicability of Leitao's findings to the mathematics classroom.

Larsen and Zandieh (2008) also explore the role of inconsistencies in the learning of mathematics. In this study, Larsen and Zandieh adapt Lakatos' (1976) framework about the process of *proofs and refutations* in the evolution of mathematics to describe student learning that occurs in classrooms. Lakatos' framework addresses the ways a person might respond to counter-examples to a conjecture or theory. Lakatos mentions that, historically, one way mathematicians dealt with counter-examples was to ignore them or to describe them as exceptions to the rule. This was similar to what Balacheff (1991) noticed in his study that was discussed above. Lakatos also describes a more

sophisticated way mathematicians deal with counterexamples, a method that he says led to mathematical discovery. He refers to that process as *proofs and refutations* and describes the four stages involved in the process.

The first stage is the development of a conjecture or theory. The second stage is the development of a proof, which Lakatos describes as a rough thought experiment where the conjecture is broken down in to smaller parts, each of which is explored. The third phase is the identification of counter-examples that contradict the conjecture. The fourth phase involves the analysis of the proof to determine which of the smaller parts of the conjecture identified in the proving phase is violated by the counter-example. The result of the analysis is a revised conjecture, featuring a new concept that was motivated by the proof analysis. In their report, Larsen and Zandieh (2008) adapt this method of mathematical creation to describe student learning.

In particular, Larsen and Zandieh (2008) saw that, in their classrooms, creating and analyzing proofs created motivation for students to revise conjectures, and generate theorems and concepts in a way that paralleled Lakatos' (1976) description of the evolution of mathematics itself. This heavily relied on students encountering and overcoming inconsistencies in their arguments. In their paper, Larsen and Zandieh give an example of a group of students writing a proof in an undergraduate group theory course. In their example, the students are trying to create a minimum list of sufficient conditions for a subset of a group to be a group. A group of students came up with a conjecture and offered a rough proof of their conjecture which relied on a faulty assumption. The teacher then presented them with a counter-example to their conjecture, which they dismissed at first. Eventually, the students analyzed both their previous proof

and the counter-example presented by the teacher. This enabled them to revise their conjecture. As the students worked on their conjecture and its related proof, they went through a process similar to the one described by Lakatos, and eventually reinvented a theorem from abstract algebra. In this way, engaging in proving helped the students to reflect on a counter-example to their conjecture, and to revise their thinking.

Larsen and Zandieh's (2008) study involves undergraduate mathematics students working on an advanced mathematical proof, but younger students can go through similar processes with less formal mathematical topics. For instance, I have seen students engage in a similar process while generating strategies for calculating answers to patterning problems (Bartlo, Chapter 3 of this document). However, future research should be done to explore if the process of *Proofs and Refutations* can create opportunities for students to learn mathematics regardless of whether they are creating formal or informal proofs.

Connections. Proof can unify and simplify theories by connecting unrelated statements. This is related to Sierpinski's (1994) idea that learning can be described as building a network. It is also related to Hiebert's (1992) definition of conceptual understanding as a connected network of ideas. In this way, proof can help students learn mathematics or develop a conceptual understanding of mathematical ideas because it can help students build a connected network of ideas. In other words, proof can promote the learning of mathematics because it can enable people to see connections and relationships between ideas. This idea appears in the proof literature in various ways. For example, several researchers argue that when students prove, they extend their existing knowledge

to create new knowledge. In fact, many of the studies that tie the role of proof to the construction of knowledge are related to this idea (e.g. Balacheff, 1991; Leita, 2000; Wood, 2001). Those studies describe how students extend their prior knowledge to construct new knowledge while engaging in proof. In a sense, those studies all appeal to the fact that proof can unify and simplify theories by connecting unrelated statements by showing how proof enables students to connect new ideas to old ideas.

Other researchers address the idea of connections more explicitly. For example, Wilkerson-Jerde and Wilensky (2011) investigate how expert mathematicians make sense of mathematical ideas that are unfamiliar to them by reading a published proof. Based on their research, they say that the proof enabled the mathematicians in their study to build connections between new ideas and old ideas. Given this, they say that mathematical knowledge can be described as “a connected network of resources that comprise... a given mathematical idea” (p. 22). Building upon Sierpinski’s (1994) previous idea that learning can be described as building a network, they conclude that proving can help students learn mathematics by helping them connect seemingly unrelated statements.

Uhlig (2002) describes how he harnesses this idea for classroom practice when describing how he teaches linear algebra. As exemplified in the organization of his textbook, he approaches teaching linear algebra by having students actively engage in proving rather than acting as passive recipients of proofs presented by the teacher. In this article, Uhlig argues that engaging students in proving promotes understanding of mathematics because it unifies all of the subject matter in the course. In this way, Uhlig is drawing on the fact that proof unifies ideas and helps students create a connected network of ideas to promote learning in the mathematics classroom.

Although they are related ideas, proof as a vehicle for connecting ideas is slightly different than proof as a means of seeing consequences of ideas. When proving enables a student to learn about the consequences of an idea, a student is learning about that idea by applying it in the proof. That is, a student can learn about a definition or theorem she uses in a proof by seeing the implications of it that are made apparent by using it in a proof. However, when proving serves as a means of connecting ideas, a student can learn new ideas by seeing how new ideas are implied by existing ideas. Also, a student can see how existing ideas are related. So although both ideas are related to implications or consequences of an idea, one function helps students learn only about an existing idea (*consequences*) while the other function lets students learn about new ideas or connect multiple old ideas (*connections*).

Global perspective. It is clearly evident that de Villiers' (1990) concepts of *inconsistencies* and *connections* are relevant to how proof can be seen to promote the learning of mathematics in the classroom. However, the *global perspective*, *application* and *alternative systems* sub-functions are less evident. These subcategories are described in more detail below.

In mathematics, proof can provide a global perspective on a topic by “exposing the underlying axiomatic structure of that topic from which all other properties may be derived” (de Villiers, 1990, p. 20). For instance, in group theory one can refer to the three components in the definition of group to derive other properties of groups. Therefore, by looking at those three axioms, one can gain a global perspective of the group concept.

While it is possible that proof could promote learning in the mathematics classroom in the same way, research has not explicitly addressed this. In fact, in studies of secondary teachers' and research mathematicians' reasons for incorporating proof in their classes, none listed the use of proof as a means of exposing the underlying axiomatic structure of a topic (Knuth, 2002c; Staples et al., 2012; Yopp, 2011). This absence may be because this is often considered the final stage of mathematical thinking (Rasmussen et al., 2005; Tall, 1992), and because few students are believed to reach a level of mathematical sophistication to appreciate such systems (Burger & Shaughnessy, 1986; Harel & Sowder, 1998). Taken together, these two arguments could suggest that few educators see using proof as a means to gain a global perspective as appropriate for their students. Additionally, since it can be seen as displaying mathematical thought rather than promoting mathematical thinking (Skemp, 1971), teachers may not see this role of proof as a practice that could be leveraged to promote learning in the mathematics classroom.

De Villiers (1986) expresses similar criticisms of deductive axiomatic approaches to teaching. However, he argues that if we teach axiomatic structures in an a posteriori manner that it could promote learning in the mathematics classroom. He references Freudenthal's (1973) argument that if students begin with specific ideas and then participate in structuring the ideas to create a more global system, then axiomatizing can support student learning of mathematics. Although this article is theoretical, research programs based on Freudenthal's ideas show evidence of how engaging in a posteriori axiomatizing can help students learn mathematics (e.g. Larsen, In Press). Consequently, although research suggests reasons for why proof as a means of providing a global

perspective may not be a productive means of promoting learning in the mathematics classroom, research also suggests how reconceiving the idea of axiomatization could create a context in which proof as a means of providing a global perspective could promote learning.

Application. Proof also helps mathematicians learn about the applicability of ideas both inside and outside of mathematics. This could be true for students of mathematics as well. Furthermore, a classroom analog to this role could exist as well. That is, proof could be seen to help students generalize their solutions so they can apply the solutions to different problems they encounter. For example, Lanin, Barker, and Townsend (2006) state that the justification process leads students to generalize their solution strategies to different problems. They argue that discussions involving justification allow students to observe how a rule applies across various cases and to construct generalizations. They also argue that it helps other students develop an understanding that enables them to generalize the solution to similar problems.

Alternative systems. In the field of mathematics, proof's role in systematizing results can lead to the development of alternative deductive systems that are more powerful or elegant than existing systems. A similar analog is true in the mathematics classroom: proving can lead to the development of new ways of thinking about problems and new solution strategies. For instance, Weber (2010) used various data to show examples of students learning new ways of thinking by discussing proofs. He provided the example of elementary school students solving a combinatorics problem. A student

used a series of zeroes and ones to represent the different possible solutions for the problem context, and used that representation to prove that he had found all of the solutions to the problem. Weber argues that, more important than solving the problem, was the realization that combinatorics problems could be solved with a binary string. In this way, the students learned a new way of thinking about combinatorics problems by justifying a solution to one combinatorics problem. In a similar manner, students can learn other mathematical ways of thinking about problems by proving. Harel (2001) shows how students can learn mathematical induction by engaging in proving, and Hanna and Barbeau (2008) argue that proofs can bring new mathematical techniques to the fore.

Summary. As argued by de Villiers (1990), proof serves as a means of systemization in the field of mathematics in that it (1) identifies inconsistencies, (2) unifies and simplifies mathematical theories, (3) provides a useful global perspective on a topic, (4) aids in application within and outside mathematics, and (5) leads to alternate deductive systems that are more elegant or powerful than existing ones. The first two of these are clearly addressed in the proof literature, perhaps in a less formal analog, and ideas related to the last two are also evident in the proof literature. However, the third subcategory is absent from the literature. In fact, a review of the literature suggests an explanation why the concept of “global perspective” may have been underappreciated. In contrast with the role of proof in mathematics research, proof is less important as a means to systemization in the mathematics classroom. Nevertheless, proof has the potential to play a role similar to that of systemization in promoting the learning of mathematics—even if that role is underappreciated.

Discovery

Proof as means of discovery⁹ is discussed in great detail in de Villiers' (1990) paper. He explains that proof serves as a means of discovery by both enabling *exploration* and *analysis*. He explains that there are many examples where ideas were discovered in a purely deductive manner (*exploration*). For example, ideas like non-Euclidean geometry could only have been created by exploration via proving. He also gives an example of a generalization of the proof that if you connect the midpoints of a kite the resulting shape is a rectangle. He explains that the proof rests on the fact that the diagonals are perpendicular, meaning that this result can easily be generalized to any quadrilateral with perpendicular diagonals. This generalization led to the creation of a new idea, not by exploration but by identification of the key idea in the proof (*analysis*). Just as proof can promote the discovery of new ideas in the field of mathematics, it can promote learning in the classroom by enabling students to discover new (to the learner) mathematical ideas.

Exploration. As is mentioned above, proving can lead to new results because it enables a person to explore the implications of a statement. In the mathematics classroom, by extension, proof can help students learn mathematics by enabling exploration of the implications of an individual's existing knowledge. Two researchers offer theories as to how proof can do this.

⁹ One could view mathematics as something that is *discovered* or as something that is *created* by individuals. I use discovery here to be consistent with de Villiers (1990), but I have combined research from each of these two views into this section.

Herbst and Balacheff (2009) point out that proofs can help students transition to a new state of knowing. Their premise is that, in the math classroom, students need to acknowledge the transition from a point in time when they are entitled not to know something, to a point in time where they are accountable for knowing it. Their work is theoretical, but they contend that the process of proving enables students to use a known idea to create or understand a new one by mapping a known conception onto a new one. They argue that this is done by using rules about known objects to justify operating on objects in a proof, which leads to a new conception. They call this new conception *proof-organized knowledge*. In other words, Herbst and Balacheff argue that students begin with their current knowledge, and extend that knowledge by connecting it to other known facts until they construct (or discover) a fact that is new to them. In this way, proving can be seen to promote learning in the mathematics classroom by motivating students to explore the implications of their existing knowledge and previously held conceptions.

Otte (1994) makes a similar claim in his paper where he discusses the paradox of the relationship between proof and the development of mathematical knowledge. He talks about knowing as “a reduction of the given to something that is already part of our mental stock” (p. 304). He says that since proof helps us connect new ideas to existing ideas, there is no new mathematical knowledge, only the extension of existing knowledge. However, I would argue that the extension of existing ideas constitutes learning. In other words, I would argue proving promotes learning in the mathematics classroom by enabling students to explore their existing knowledge and, in turn, extend it to “new” knowledge.

The role of proof described in these two papers is similar to the role proof plays in establishing connections between ideas. However, when proof serves as a means of exploration the focus is on discovering a new idea by exploring the implications of an existing idea. In establishing connections between ideas the idea is to connect a new idea with an old one.

Both of the papers discussed above are theoretical. Although empirical research has not explicitly addressed this topic, previous research can be interpreted to support the hypothesis that proof can promote the learning of mathematics by enabling students to explore and extend their existing knowledge. For instance, in a study of second graders, Wood (2001) showed that students rely on their previous experiences when they encounter new mathematical experiences. When proving, students often build from one idea to another. Given that students address new tasks by starting from their existing knowledge, their only recourse would be to start with their existing knowledge and apply or extend that knowledge to discover new (to the learner) knowledge. In other words, a classroom culture based on proof and justification can be seen to create opportunities to learn new (to the students) mathematical ideas by enabling students to explore the consequences of their prior knowledge.

Analysis. In addition to promoting the discovery of mathematics by enabling exploration, proof can also lead to the discovery of mathematics because, at times, the key idea of an argument can be generalized to make a stronger statement than the one that is being proved. That is, by analyzing a proof, an individual can identify the essence of the proof, or the key idea (Raman, 2003) of the proof. Once the key idea is identified, the

proof can be generalized to any objects that meet the constraints of the key idea. So analysis of a proof can lead to the discovery of mathematics because it can help mathematicians generalize their results and it can motivate the need for new concepts. Proof can promote the learning of mathematics in a similar way in the classroom as well.

Proof can promote the learning of mathematics by helping students to learn as they reflect on the key idea of an argument in another way as well. That is, that after identifying the key idea of an argument, a student may feel the need to investigate the key idea. Kidron and Dreyfus (2010) address this idea when they describe a solitary learner's experience with justification as she investigated bifurcation points in dynamic systems. In their study, they observed that the learner felt a need to justify because she wanted to gain insight into a bifurcation point. As she engaged in justification, she experienced "a sequence of three degrees of enlightenment" (p. 89). They argue that these three stages of enlightenment led to her understanding her object of study. They note that there is "a strong relationship between combining constructing actions and justification as enlightenment" (p. 90). The enlightenment they describe came from reflecting on the key ideas of the justification. This means that their study showed that reflecting on the key ideas of the justification led to learning about the mathematical content at hand.

Summary. Discovery is another important consequence of proof. Much like in the field of mathematics, by using an idea in a proof, students can learn about the consequences of that idea. In this way, students may discover a new concept or idea by engaging in proof. This is connected to many of the ideas discussed by researchers in relation to learning through proving. For example, the researchers who focus on the

construction of knowledge (Balacheff, 1991; Herbst & Balacheff, 2009; Leita0, 2000) show how engaging in proof helps students construct new (to the learner) knowledge by extending their existing knowledge. The idea that proving motivates a need to reflect on ideas (Harel, 2001; Larsen & Zandieh, 2008; Weber, 2010; Yackel, 2002) is also connected to the idea of discovery because proving is seen to motivate analysis of ideas. Proof leads to discovery in the same ways in the field of mathematics. Sometimes mathematicians discover new mathematical ideas by exploring the consequences of assumptions (as pointed out in the first group of articles). At other times they first recognize the need to invent a new concept while analyzing their ideas in the course of constructing a proof (as pointed out in the second group of articles).

Communication

De Villiers (1990) mentions that, in the past few decades, the communicative function of proof has been getting increased attention. Proof serves as a means of communication in two ways: it represents a unique way of communicating mathematical results within the mathematics community, and it also creates a forum for debate in the field of mathematics. Proof can be seen as a social process of sharing mathematical ideas—a way of communicating ideas within the mathematics community, or a *form of discourse* within the mathematics community. In addition, since proof serves as a means of verifying ideas, while communicating ideas with one another via proof, proof plays a regulatory role that facilitates the avoidance of mathematical mistakes. In this way, proof can be seen to create a *forum for debate*. Just as proof serves as a form of discourse and a forum for debate in the field of mathematics, it can serve those same roles in the

classroom. Additionally, the fact that it plays those roles can be leveraged to serve other classroom purposes as well.

Form of discourse. Recent reform efforts have focused on the importance of communication in the mathematics classroom (e.g. Common Core State Standards Initiative, 2009; National Council of Teachers of Mathematics, 2000). This is based upon the belief that teaching students to communicate in mathematics classes will help them to clarify their thinking and increase their understanding. Proofs are intricately related to communication of mathematical ideas. They are the format through which mathematicians share their ideas with one another, and the same can be true within a classroom. By creating proofs, students are communicating their ideas about a topic. Because proving puts students' ideas on display for the teacher and for other students, proving can promote the learning of mathematics in the classroom.

When students' ideas are put on display for the teacher, this can promote learning by enabling teachers to assess student understanding. Students' justifications provide a window into their understanding (Lannin, 2005; Vanderhye & Zmijewski, 2007). So, when students justify their answers, teachers have an opportunity to identify gaps in student understanding. Teachers can leverage these insights to modify instruction to meet the needs of their students. Flores (2002) argues that, to gain insight into students' thinking, we must ask them to justify what they are doing. He does not give a reason for why we need to ask them to justify in order to gain insight into their thinking, but does provide examples of students' justifications and what those justifications show the teacher with respect to the students' understandings. In particular, his examples involve

students' strategies for solving word problems involving addition. His examples illustrate the potential for teachers to learn how students went about solving such problems.

Researchers have also shown that teachers explicitly recognize this as one of the purposes of justification in their classrooms. During interviews and work sessions centered on the topic of justification, middle school teachers identified the opportunity to learn what their students understood as one of the reasons why they incorporate proof in their classrooms (Staples & Truxaw, 2009; Thanheiser, Staples, Bartlo, Heim, & Sitomer, 2010). Like reports from Lannin (2005) and Flores (2002), this report does not offer any explanation as to how justification helps to achieve this goal.

There is, however, ample evidence in the research literature showing that researchers can learn about their research participants' understanding by asking them to engage in proving activities. For instance, several researchers have given examples of things they thought their research participants understood, which they realized the participants did not understand after the participants used the ideas in proofs. In particular, the researchers realized that the participants did not understand the definitions in the ways the researchers had intended by observing how participants applied the definitions in proofs, (e.g. Alibert & Thomas, 1991; Ball & Bass, 2003; B. Edwards & Ward, 2004).

Another way that putting students' ideas on display can promote learning is that it gives students an opportunity to consider one another's justifications. Many researchers argue that opportunities for learning arise during argumentation as students attempt to make sense of justifications given by others (e.g. Weber, Maher, Powell, & Lee, 2008;

Whitenack & Knipping, 2002; Wood, 1996, 1999; Yackel & Cobb, 1996). Researchers argue that this is because reflecting on the thinking of others allows students to make connections between the different ways students model their activity.

Additionally, sharing proofs can create learning opportunities for the student sharing the proof as well. For instance, Whitenack and Yackel (2002) argue that when students share their proofs they can revisit their mathematical ideas, creating opportunities for them to refine their ideas. Moreover, Whitenack and Yackel argue that as students respond to challenges from their classmates they may build stronger mathematical arguments or find new ways of looking at the problem. This can help students develop a more thorough understanding of the ideas they are grappling with and can help them construct new understandings.

Although researchers contend that students' engagement in argumentation creates opportunities for learning, researchers also stipulate that the interrelationship between argumentation and learning is only beginning to be understood (e.g. Wood, 1996, 1999). In particular, one question that could be raised is whether the same learning opportunities occur when students share their solution strategies without justifying their solutions. Related research would suggest this is not the case. An analysis of several elementary classrooms showed that students exhibit more complex thinking in classrooms where students engage in argumentation as compared to classrooms where students simply report their solution strategies (Wood, Williams, & McNeal, 2006). Similarly, Cobb, Wood, Yackel, and McNeal (1992) report that when students are engaged in proving, they have opportunities to learn about mathematical concepts and ideas. In contrast, according to their study, when students are not engaged in proving, they are likely to

learn only procedures. In this study, Cobb et al. analyzed the learning that occurred in second and third grade classrooms with different classroom norms. When students were not asked to justify their results, learning involved enacting procedures successfully. In classrooms where students engaged in justification, the students were able to navigate problem situations in a more meaningful way. The findings of this study suggest that engaging in justification helps students develop conceptual, rather than merely procedural understanding. In this way, it suggests that students may learn more by sharing their ideas through proving than by using other forms of discourse.

Kazemi and Stipek (2001) also observed that when students' explanations consisted of mathematical arguments, they were more likely to develop conceptual understanding than in other situations. That is, by looking at fourth and fifth grade students, Kazemi and Stipek observed that justifying answers motivated students to think conceptually rather than procedurally. In particular, Kazemi and Stipek (2001) argued that justifying promotes learning of mathematics when students compare and contrast solutions. The four classes Kazemi and Stipek studied were all inquiry-oriented classrooms with many of the same social norms. In classrooms that promoted conceptual thinking, students were expected to give a mathematical argument as to why their solution worked. In contrast, in other classes, students described or explained the procedures they used without arguing why their solutions worked. Therefore, this study also suggests that communicating ideas via justification of solutions is an important part of helping students to develop conceptual understanding in the mathematics classroom.

The use of proof as a form of discourse in the classrooms can be different from its use in research mathematics. In research mathematics it is accepted that published proofs

appear in formal, symbolic, deductive forms. This may be the case in advanced mathematics classes, but in earlier mathematics classes this is most likely not the case. In various math classes, proofs may appear in various forms, including informal, narrative, and pictorial formats. Formal proofs may seem like a foreign language to many mathematics students—a practice to which they do not have access. In contrast, those students may view justifying and arguing as accepted social norms that are accessible to them. Although the discourse may appear in different forms in the two communities, the role of proof as a form of discourse in both communities are still related.

Additionally, in general, mathematicians do not look to each other's proofs to assess one another's understanding of ideas. In the field of mathematics, the communication aspect of proof is primarily related to learning about why something is true or how a mathematician thinks about an idea, rather than to assessment. That said, mathematicians do often look to one another's proofs to learn different solution strategies. So there are some similarities and some differences between the communicative purposes proof plays in the field of mathematics and in the mathematics classroom. Given the different goals of the two communities, it is not surprising that proof serves a slightly different communicative function in the two communities.

Forum for debate. Putting students' ideas on display has another result, too: it can create conflict for the students to resolve. When conflicts arise, students need to provide justifications for their actions or for their challenges to the ideas of others. This means that students settle their debates via proving. In other words, by serving as a form

of discourse, proof can also serve as a forum for debate. Engaging in a forum for debate promotes learning in the mathematics classroom in several ways.

Some of the ways this can happen are discussed in detail in previous sections. For instance, this is related to dealing with inconsistencies. When conflict arises as students share and debate their ideas, students often have to deal with inconsistencies, circular arguments, or implicit assumptions in their thinking. As students respond to those conflicts, they often have to revise or refine their thinking. Another way engaging in a forum for debate can promote learning is related to the idea of verification. When students are convinced of the soundness of previous arguments, ideas cease to be challenged when used in a future argument and become taken-as-shared within the classroom community. So having a forum for debate (created by students engaging in argumentation) can both help students reach resolution about an idea and promote learning in the mathematics classroom.

An emphasis on argumentation can also promote learning in the mathematics classroom by creating opportunities for new mathematical concepts and tools to emerge. Yackel (2002) illustrates this claim by using an example from an undergraduate mathematics classroom. In that classroom, students were expected to provide justification for any claims they made, and those justifications were open to be challenged by other members of the classroom community. In her example, new mathematical ideas and tools needed to be developed for students to justify their original reasoning in response to a challenge from other students. Therefore, when students provided support for or against their claims, they created opportunities to learn new

mathematical ideas. Consequently, the author shows that argumentation may promote learning of mathematics by inspiring students to make their ideas more explicit.

Larsen and Zandieh (2008) make a similar claim: that engaging in proving can motivate students to construct new mathematical ideas. As was described previously, Larsen and Zandieh adapted Lakatos' (1976) framework about the process of *proofs and refutations* in the evolution of mathematics to describe student learning that occurs in classrooms. In particular, Larsen and Zandieh (2008) observed that creating and analyzing proofs created motivation for students to revise conjectures and generate theorems and concepts in a way that paralleled Lakatos' (1976) description of the evolution of mathematics itself. In this way, proving created a forum for debate, which helped students learn mathematical concepts.

Not only is there a norm that answers are justified in the field of mathematics, there are also established criteria for determining what counts as appropriate justification. The same is true in the mathematics classroom. These criteria, or sociomathematical norms (Yackel & Cobb, 1996), guide how ideas are justified and how conflicts are resolved in the mathematics classroom. Although the form in which proofs are presented in classrooms can be less formal than in the field of mathematics, proofs still need to be based on mathematically valid forms of reasoning. That is, debates must still be settled in mathematically legitimate ways. So the rules that govern the forum of debate may differ between the mathematics community and the classroom community, but the general idea is still the same. Just as mathematical proof is how the mathematics community determines if claims are true and arguments are valid, the same principle

applies in the mathematics classrooms. Student learning can be fostered when students engage in a process similar to the process in which mathematicians engage.

Summary. Proof provides a means for students to communicate with their peers and their teachers. By displaying student thinking or understanding, proof allows students to learn from one another and also for teachers to learn about their students (thereby promoting student learning). Proof provides this opportunity because it serves as a means of communication by functioning as form of discourse and creating a forum for debate. Although this is true in the field of mathematics and in the mathematics classroom, they function slightly differently in the two communities because the two communities have different goals.

Several questions remain such as: does proof play special communicative roles in the mathematics classroom, and, if so, what are those roles? Additionally, much of the communicative roles described above are based on the transference of information, and the question remains whether proof plays a special role in ascertaining this information. It seems likely that proof does play a special communicative role in the mathematics classroom. The field would benefit from future research in to what is unique about proof as a form of communication in the classroom. In particular, when a student justifies her answer, she explains what actions she took and why she took those actions. Consequently, students and teachers do not need to speculate as to what actions the student took or why she took them. This provides students and teachers with more information about the student's thinking than if she had simply stated an answer or explained her solution process.

Discussion

Consideration of the role of proof in the field of mathematics provides a good starting point for thinking about the role it can play in the classroom. Proof can promote learning of mathematics in the classroom community in much of the same way it does in the mathematician community. However, relying solely on the roles proof plays in mathematics to describe the roles it plays in the classroom is limiting for several reasons. First, teachers' goals in the classroom are not identical to the goals of mathematicians conducting research (Staples et al., 2012). Second, children's thinking may not follow the same structure as that of experienced mathematicians (Wood, 2001). Third, if we "pigeon-hole" the roles of proof into one of the existing categories derived from analysis of mathematics research, we may overlook unique roles proof can play in the classroom (Yopp, 2011).

Abandoning the role proof plays in mathematics when considering the role proof plays in the classroom has consequences too. One consequence is that some of the important aspects of the field of mathematics may be hidden in the discussion. For instance, if you say the role of proof in mathematics is explanation, it begs the question of the importance of using proof at all, why not just provide explanations. One of the responses to this is that by the nature of mathematics proof offers special forms of explanation, such as connecting ideas and deepening existing knowledge, as were discussed in the previous sections. Connecting the idea of explanation to the explanatory role proof plays in the field of mathematics offers insight into the explanatory role proof can play in the classroom.

A related example is that of displaying student thinking. Studies have shown that teachers do not cite communication as a reason for using proof in the classroom, but that teachers do use it to display student thinking (Knuth, 2002c; Yopp, 2011). However, in the field of mathematics proof is a form of discourse, meaning it is the way mathematicians share their thinking or their reasons for believing a claim is true with one or another. So proof's role in displaying student thinking is tightly connected to its role of communication in the field of mathematics. Separating the two ideas hides the important role proof plays in mathematics, and it hides the reasons why proof is a particularly good vehicle for displaying student thinking.

In these ways, using only de Villiers (1990) framework can be a limiting way to discuss the role of proof in the classroom, but abandoning it can hide potentially valuable insights and perspectives. Consequently, expanding de Villiers' framework to show both the ways proof can promote learning in the field of mathematics and the specific ways teachers harness those roles for their educational goals in the classroom has the potential to be faithful to both classroom practices and mathematical practices. The framework I offer in this paper shows how proof can play the same roles in the classroom as it does in the field of mathematics, and it also shows how the roles of proof can be leveraged to reach other important instructional goals.

Another reason for connecting the roles proof plays in the classroom to the roles it plays in the field of mathematics is that it highlights both the similarities and the differences between the role proof can play in the two communities. The differences between the roles proof plays in the two communities are nuanced, and can easily be hidden by the similarities between the practices of the two communities. One of the

differences between the two communities is emphasis. That is, in the mathematics classroom certain practices, such as explanation, are particularly valued, while in the field of mathematics verification is particularly important. Additionally, proof's role in communication is so ubiquitous it almost goes unnoticed in the mathematician community, while it is more obvious in the classroom because students often have to learn how to participate in the practice of proof at the same time as they are using it to communicate. Highlighting the similarities and the differences between the communities allows mathematics educators to better understand the role proof can and does play in the classroom.

Implications

Thinking about the specific roles proof can play in the classroom as well as thinking about how it plays those roles has several implications for instruction. For instance, it is valuable for curriculum designers and for instructors as it offers insight as to when to ask for justification. In an activity there are many opportunities to push students, but if all of them are capitalized on an activity may never progress forward. Teachers need to select which of the openings they will press on. Thinking about the role proof plays in a given lesson can help teachers and curriculum designers choose those times because it helps them identify what their goal in the activity is, which can help them determine what to press for in their classes. Additionally, this framework gives a way of understanding why different teachers might implement the same lesson very differently. Understanding the different goals that can be accomplished by using proofs can give language to discuss different implementation choices and different educational

outcomes. This could be useful for the design of instructor support materials and for shared lesson planning. It can also be useful for researchers studying teachers, to understand the nuances between different actions different teachers take.

This framework highlights many interesting relationships between the roles proof can be play in the classroom. For instance, if the discussion is about using proof as a means of communication, this chart describes different ways proof can do that. However, there are several similarities between the ideas represented in the framework that are not highlighted by this organization. For instance several of the subcategories relate to developing understanding or to developing new ways of thinking. Consequently, if a teacher, researcher, or curriculum designer wanted to focus on how proof could serve one of these purposes, it could be useful to cluster several of the subcategories listed in this framework together, in order to highlight the different ways proof could play the given role. *Figure 2* (below) shows two clusters of the ideas discussed in this framework that might be useful for certain teacher's, researcher's, or curriculum designer's goals.

#1 Understanding	#2 New ways of thinking
Insight	Application
Consequences	Alternative Systems
Exploration	Exploration
Analysis	Analysis
Connections	
Global Perspective	

Figure 2. Clusters that highlight connections between roles in the framework.

The first grouping organizes all of the functions related to showing why a mathematical concept is true, how the concept works, or how the concept is related to other ideas. These subcategories come from the categories of *Explanation*, *Systemization*, and *Discovery*. They begin with the broad notion that engaging in proving can provide insight into why the mathematical phenomenon being proven is true and how it works (*Insight*). The later subcategories offer more specifics about how that can happen. These begin with the idea that a person can learn about an idea when it is used in a proof because applying the concept in a proof shows what the consequences of that idea are (*Consequences*). On the other hand, engaging in proving can help students learn new ideas because they can see what new ideas are consequences of an existing idea (*Exploration*). Similarly, by generalizing the key idea of a proof, a student can extend an existing idea and discover a new related idea (*Analysis*). Engaging in proving can also help students connect unrelated statements (*Connections*). Finally, proving can help students gain a global perspective of a topic, which relates to the ideas of unifying theories and building a connected network of ideas (*Global Perspective*). Looking at these ideas together can help highlight the nuances for the different ways proving can help students develop an understanding of mathematical topics.

The second group is about how proving can help students develop new ways of thinking. These subcategories come from proof as a means of *Systemization* and proof as a means of *Discovery*. Proof can help students generalize solutions to other problems (*Application*). Proving can also help students learn new methods for solving problems (*Alternative Systems*). Engaging in proving can also help students discover new methods and strategies for solving problems as they explore consequences of concepts

(*Exploration*) or as the generalize strategies or key ideas from their proofs (*Analysis*).

This group of subcategories highlights the ways in which proof can help students learn new ways of thinking about mathematics problems.

It is worth mentioning that the roles described in the may not be completely distinct. For instance, if a teacher's goal is for students to uncover inconsistencies in their thinking, she will likely also rely on proof's role as a forum for debate and as a form of discourse. Additionally, a teacher may use a task for a specific purpose, but it may actually serve another purpose at the same time. Similarly, a teacher might have more than one role of proof in mind, and could use a task to serve more than one role.

Ideas for Future Research

The reframing of de Villiers (1990) framework presented in the body of this paper is based on the connections between proof and learning mathematics that are purported in the literature. However, it is likely that this is not an exhaustive list of the ways experts believe proof promotes learning in the mathematics classroom. This is a consequence, at least in part, of the fact that the vast majority of research on the topic implicitly assumes that proof promotes the learning of mathematics without any need for elaboration. That is, many researchers simply assume that students develop deeper or more meaningful knowledge in classrooms where proving is a prominent practice as compared to classrooms where it is not (e.g. Ball & Bass, 2003). They do not provide any explicit insights into how proving leads to those outcomes. For research to advance in the field, researchers need to become more explicit about the underlying mechanisms whereby proof promotes the learning of mathematics.

One limitation of this study is that it only draws on literature that was uncovered through the bibliographic search that I described in the beginning of this paper. It is highly likely there is literature related to these ideas that were not uncovered by the search for various reasons including that they did not use any of the keywords for which I searched. Consequently, this framework should be considered an emerging framework as it serves as a starting point for describing the connections between proof and learning that are discussed in the literature and is not a complete survey of all of the possible connections between the ideas.

It is also worth noting that this synthesis draws on literature covering a myriad of conceptions of proof (e.g. ranging from formal to informal arguments and from the process of proving to finished products). Researchers have argued that not all proofs are equally suited to develop conceptual understanding (e.g. Weber, 2005). Similarly, this synthesis draws on studies done at different grade levels. Research shows that proofs can look different at different grade levels, and that different age students engage in different types of thinking and reasoning. This begs the questions: Are all types of proofs equally suited to achieve all of the functions discussed in this paper? And do all of these roles make sense at all grade levels? Clearly, more research needs to be conducted to assess the generalizability of these claims.

Many of the papers described in this synthesis offer theoretical accounts of the roles proof could play in the classroom. Researchers often conjecture why proof leads to the outcome they mention, but very few use data to show how proof might lead to those outcomes. Empirical research should be done to explore these theories. Many papers draw on empirical evidence and mention a connection between engagement in proof and

the purported outcome. One study not previously discussed in this synthesis suggests a correlation between emphasis on proving and student outcomes on standardized tests, but does not further elaborate on the connection (Boaler & Staples, 2008). In other words, even the empirically based studies do not offer insight as to *how* proof promotes the learning of mathematics. This suggests further qualitative research in this area is needed.

Another gap in the research relates to how proof uniquely contributes to these learning outcomes. More research needs to be done to explore this issue, and to explore if there are more ways proof leads to the learning of mathematics. This is not to suggest that proof does not have value as a learning practice even in the cases where proving may not make a unique contribution to create learning opportunities. Since proving has other benefits, one might choose to use proof as a way to reach a learning outcome if one believed that proof has potential to serve multiple roles at the same time. However, it may be that it has unique properties that make it essential as a classroom practice.

This review also neglects the literature addressing other reasons educators incorporate proof in their classrooms. That is, proof is a practice in mathematics classrooms for reasons other than to learn the concept of proof or to promote the learning of other concepts. For instance, some educators use proof to help students learn critical thinking, to reach affective goals, or to develop specific habits of minds. That research was not discussed in this chapter because it is beyond the scope of the paper, but it is worth mentioning as it is tangentially related to the topic at hand. For instance, in the discussion of the verification role of proof I mentioned the idea of intellectual autonomy and how the development of intellectual autonomy could foster student learning. Since proof can be seen to help develop intellectual autonomy, proof can be seen to promote

learning through developing autonomy. Similarly, researchers have addressed the role of proof in developing metacognition (Lannin et al., 2006), developing higher level thinking skills (Ball & Bass, 2003; Fawcett, 1938; Forman, Larreamendy-Joerns, Stein, & Brown, 1998), and learning to treat each other with respect (Boaler, 2004, 2008). Since one could argue that traits such as these could help promote the learning of mathematics in the classroom, one could argue that fostering these traits promotes learning in the mathematics classroom as well. Therefore, exploring the relationship between proof, developing habits of mind, and learning mathematics could be another fruitful avenue to further research on the role of proof in learning mathematics.

Although the literature purports a wide variety of roles proof could play in the mathematics classroom, research on teachers' ideas about proof has not shown the same diversity. In Knuth's (2002b, 2002c) study with secondary teachers, he observed that the teachers he interviewed thought that the primary purpose of proof in the classroom was verification. He saw little evidence that they thought proof could help promote understating. In another study involving middle grades teachers, researchers report that the teachers they interviewed viewed the main purposes of justification in the classroom to help students learn for themselves and from each other, and also to provide the teacher with information about the students' thinking and understanding (Staples et al., 2012; Thanheiser et al., 2010). In a study of university level mathematics professors, Yopp (2011) reported that the main reason the professors he interviewed used proof in lower level courses was to teach students how to prove. In the courses where students were comfortable with proving, the professors described using proof for a variety of reasons such as to verify statements for students, to explain statements to students, to help

students organize ideas into an axiomatic system, and to help students learn about math. The professors did not mention using proof to facilitate communication or exploration in their classrooms.

In all of these studies, the interviewers asked the teachers about their thoughts on the role of proof (in the case of Knuth and Yopp) or justification (in the case of Thanheiser et al.) in the classroom, but did not ask their opinions about the roles other teacher's mentioned but were not mentioned in the interviews. So the researchers cannot say whether or not the teachers would agree with the other roles reported in this paper. Nevertheless, these studies do suggest that teachers' conceptions include many, but not all of the ideas expressed in the literature. These studies also raise the question as to whether or not teachers at different grade levels may have different views about the role proof could serve in the mathematics classroom. These two points raise important questions. For example, what roles of proof would the mathematics education community like teachers to consider that they may not have already considered? Which views about proof might be most helpful when teaching from specific curricula (or which curricula might be most helpful for teachers with a given viewpoint on the role of proof)? Should proof serve the same roles at all grade levels and in all classrooms? The field would benefit from further examination of all these questions.

It is essential to consider the impact of the individual teacher as well as the specific mathematical problems selected for use in the classroom. The outcomes discussed in this paper might depend upon a teacher's actions or choice of problem. Future research should be done regarding which classroom practices and which teachers' actions help proof serve given roles, as well as what characteristics of mathematical

problems best facilitate this learning. In sum, this literature review serves as a starting point for discussing the roles proof can play in learning mathematics, but it also raises many questions for future research.

Finally, the relationship between argumentation and knowledge construction is currently a hot topic in the field of science education (Aydeniz, Pabuccu, Cetin, & Kaya, 2012; Ogan-Bekiroglu & Eskin, 2012). It would be interesting to see whether these discussions carry over to mathematics education. By assessing the differences between the roles of scientific argumentation and mathematical argumentation in promoting classroom learning, one might gain insight into the unique ways mathematical proof can promote learning in the mathematics classroom.

Conclusion

This literature review can serve as a first step towards establishing a framework for the role of proof in learning mathematics and it raises many questions that will help further research in this area. When talking about the promotion of *mathematics for all*, Martin (2003) argues that we should be wary of trying to promote something as a community until we have figured out precisely what it is we are trying to promote. That is, before we try to get *there*, we should figure out where *there* is. The same issue is true with proof. As a field, we seem to be saying we should incorporate proof into all classrooms, but this movement is doomed to fail if we do not articulate what this means. This literature review is intended to serve as a starting point for discussing what incorporating proof into the mathematics classroom as a learning practice might mean.

This type of theoretical framework can help us to articulate a myriad of views, thereby advancing the field in many ways. It can help researchers to understand how their views differ from the views of others, which can help move the field forward by enabling researchers to better explicate the perspective from which their research was conducted. This can help facilitate conversations between researchers and facilitate collaboration. This framework can also help curriculum designers think about how to best explicate the role proof serves in their curricula, which can, in turn, help teachers implement curricula in a manner closer to the one intended. This can help curricula be more effective, and improve student learning.

In addition to helping researchers and curriculum designers, this framework can also be helpful for teachers. It can help teachers broaden their perspectives about the roles proof can play in their classrooms. This can help teachers think carefully about their instructional goals and their instructional moves, which can improve instruction. This framework can also help teacher educators think about what goals they have for the pre-service and in-service teachers with whom they work. It can do this by giving teacher educators a framework from which to think about what viewpoints they want to help teachers develop with regard to the role of proof in their classrooms.

The analysis related to this emerging framework is a valuable starting point for conversations about the ways in which proof serves as a learning practice in the classroom. Such a discussion is important in order to move the field forward with the respect to the role of proof in the mathematics classroom. Consequently, this emerging framework and continuing to answer the questions raised by the development of this framework has the potential to advance the field in many ways.

Chapter 3

An Empirical Study

Learning Opportunities Created by Engaging in Proof: How Identifying the Key Idea of a Proof can Promote Learning in the Mathematics Classroom

In 1990 Lampert concluded her seminal work by observing that she had shown that students could engage in activities such as proving, but “the problem of defining what knowledge they have acquired remains” (p. 59). In the twenty years since Lampert’s article was published, very little research has been directed toward this end. The goal of this article is to address this issue.

It is widely accepted within the mathematics education community that proof is an important part of the mathematics classroom, but research has mainly focused on students’ understanding of proof, students’ ability to construct proofs, and ways to teach proof (Hanna & Barbreau, 2008). Hanna and de Villiers (2012) observe that proof can be seen as more than a procedure to learn, it can also be seen as a sequence of ideas and insights, suggesting that it is a process that can lead to mathematical understanding. In other words, proof can be seen as more than content to teach, but as a *process* that can lead to mathematical understanding (Bartlo, Chapter 1 of this document; Reid, 2011). If proof is to be used as a way of promoting understanding rather than as content to learn, research needs to explore the ways in which proof can promote understanding.

The goal of this article is to explore the ways in which engaging in proof can help students learn mathematical content. To do that, I examine opportunities for learning that occur in middle school mathematics classrooms where teachers value proof as a learning process. I observed that as the students engaged in proving activities, opportunities for learning occurred as students identified the key ideas in proofs. In some cases, identifying the key idea of a proof motivated the students to notice a general mathematical relationship. In other cases, identifying the key idea helped students solve other problems they encountered, or inspired them to explore related mathematical ideas.

In this paper I first describe the study itself, including the project from which the data are drawn. I then describe the learning opportunities that proving activity created in the participating teachers' classrooms, and discuss the role proving played in creating those learning opportunities. I also place these findings in the context of previous research on proof and the learning of mathematics. Finally, I conclude by offering some ideas for future research suggested by this study.

Related Literature

There is no clear consensus as to what is meant by the term proof. It is used in a myriad of ways, ranging from logical deductive chains to explanations accepted by a community at a given time (Stylianides, 2005). When referring to proof, I mean *removing doubts about a claim* (Davis, 1986), which generally involves articulating evidence and warrants in support of a claim to convince oneself or others (Toulmin,

1958). The idea of proof could involve the finished object itself (i.e. a proof) or the process of removing doubts (i.e. the activity of proving).

Discussions of the potential role of proof in the classroom often draw on its uses in research mathematics (Yopp, 2011). One of the most commonly used descriptions of the role of proof in the field of mathematics was written by de Villiers (1990). The five roles of proof de Villiers describes are: (i) proof as a means of *verification*, (ii) proof as a means of *explanation*, (iii) proof as a means of *systematization*, (iv) proof as a means of *discovery* and (v) proof as a means of *communication*. Although many other papers have been written about the role of proof in mathematics (e.g. Hanna, 2000; Jaffe, 1997; Thurston, 1995), most of the functions mentioned in those articles could be characterized by one or more of the roles described by de Villiers (Yopp, 2011). In addition to theorizing about the role proof plays in the field of mathematics, researchers have also started to look at mathematicians' practice with respect to proof. For example, researchers have begun to look at what mathematicians hope to gain from reading proofs (e.g. Weber, 2008), how mathematicians use proofs in their classes (e.g. Yopp, 2011), and what features mathematicians believe make proofs useful for pedagogical purposes (e.g. Lai, Weber, & Mejia-Ramos, 2012).

In addition to looking at the role proof plays in the field of mathematics and the practice of mathematicians to learn about the role proof can play in the classroom, researchers have also begun to look to K-12 classrooms and teachers to learn about the role proof can play in the classroom. Researchers have observed that teachers purport to use proof for many, but not all, of the purposes that mathematicians do (e.g. Knuth, 2002c; Staples et al., 2012). However, the purposes often look slightly different in

classrooms than in the field of research mathematics (Bartlo, Chapter 2 of this document). Researchers also note additional purposes such as developing critical thinking (e.g. Fawcett, 1938), and promoting equity (Boaler & Staples, 2008).

Additionally, researchers have begun to focus not just on the role proof can play in the classroom in general, but specifically at the role proof can play in the promotion of learning other mathematical content (Reid, 2011). Researchers have discussed issues such as how proofs can help students develop new ways of thinking (Harel, 2001; Weber, 2010), how the proving process relates to learning theories (e.g. Balacheff, 1991; Wood, 2001), and how proving allows teachers to learn about students' understandings (e.g. Flores, 2002; Staples et al., 2012).

Since learning can be seen as the creation or discovery of mathematics that is new to the learner, one aspect of the role of proof in the field of mathematics that it is particularly relevant to the promotion of learning in the classroom is proof's role in aiding in the discovery of new mathematics. De Villiers (1990) argues that there are two ways in which proof can lead to discovery in mathematics. One way is through deduction; de Villiers uses the example of non-Euclidean geometry, which he argues could only have been discovered in a purely deductive manner. The other way de Villiers explains that proof aids in the discovery of mathematics is through the generalization of the key idea of an argument. One example he gives of this is a proof of the statement "if you connect the midpoints of a kite the resulting shape is a rectangle." He explains that the proof rests on the fact that the diagonals are perpendicular, which means that this result can easily be generalized to any quadrilateral with perpendicular diagonals. De

Villiers argues that the identification of the key idea in this proof led to the discovery of a new idea.

In this last example de Villiers (1990) is focusing on the importance of the key idea of a proof. The notion of *key idea* is discussed elsewhere in the proof literature as well (e.g. Raman, 2003). Raman describes the *key idea* as the essence of a proof, something that promotes understanding and conviction. She adds that a proof may have more than one key idea in it. Research on key ideas often focuses on the role of key ideas in proof production (e.g. Larsen & Zandieh, 2008; Raman, 2003), but they can be an important part of making sense of presented proofs (which often involves re-creating the proof for oneself).

Kidron and Dreyfus' (2010) implicitly address the notion of key ideas in their discussion of how proofs promote the understanding of mathematics during a solitary learner's investigation of bifurcation points in dynamic systems. In their study, Kidron and Dreyfus observed that, after reflecting on the key idea of a proof, the learner felt a need to justify because she wanted to gain insight into a bifurcation point. They observed that as the learner engaged in justification, she uncovered three particularly important parts of the argument, which led her to experience "a sequence of three degrees of enlightenment" (p. 89). Kidron and Dreyfus argue that these three stages of enlightenment led to her understanding her object of study. So, they note that there is "a strong relationship between combining constructing actions and justification as enlightenment" (p. 90).

These two studies (de Villiers, 1990; Kidron & Dreyfus, 2010) focus on the importance of reflecting upon the key idea of a proof to promote understanding of

mathematics. In particular, the first article focuses on how key ideas can promote understanding in the research mathematics community by promoting the generalization of the concept being proven. The second article addresses how key ideas can motivate a search for understanding of the concept being proven. In what follows in this paper, I will elaborate on ways in which identifying key ideas in proofs can promote the learning of mathematics in the classroom. In particular, I will discuss classroom episodes where proving promotes the learning of mathematics because students identify key aspects of proofs, and, in turn, generalize, apply, or reflect on them.

Method

The data used for this study are drawn from a larger project called *Justification and Argumentation: Growing Understanding of Algebraic Reasoning*¹⁰ (JAGUAR), a study focusing on proof in middle grades mathematics classrooms. In particular, this project explored how teachers develop knowledge about algebraic justification and transform this knowledge to classroom practice. The project involved a two-year commitment from a group of participating teachers, which included attending two one-week summer courses (25 hours each), participating in six Saturday work sessions (5 hours each), participating in interviews and assessments, and conducting 8 lesson cycles. Each lesson cycle entailed planning and teaching 1- to 3-day lessons in their own classroom designed to promote justification, reviewing videos of the lessons, and writing

¹⁰ The JAGUAR project is a grant funded by the National Science Foundation (NSF) (DRL-0814829). The views expressed in this dissertation are those of the author and do not necessarily reflect the views of the NSF.

guided journal responses. The data corpus for the study discussed in this article was generated in the lesson cycles.

The Classrooms Involved in the Study

Twelve middle school mathematics teachers, drawn from five districts across two states, participated in the JAGUAR project. The teachers' professional experience ranged from 2– 30 years, and all were fully certified to teach mathematics in middle school (four held secondary credentials). The set of schools in which these teachers taught was very diverse; it included schools that serve urban, suburban, and rural communities. It is from these teachers' classrooms that the data for this study were drawn.

These teachers were selected to participate in the project largely because they already expected their students to participate in mathematical classroom discourse, and because they valued the role of justification as a learning practice in their classrooms. Research has shown that practices such as these are not widespread in mathematics classrooms in the United States (Jacobs et al., 2006). In particular, research indicates that in most mathematics classrooms students do not typically share their solution strategies or justify their answers. Thus, the teachers who participated in the JAGUAR project are not representative of the larger population of middle school mathematics teachers. Rather, the teachers who participated in this study form a *purposive* sample (Yin, 2006), chosen to enhance the investigators' ability to examine the role of proof in middle grades mathematics classrooms. In order to explore how proof may be used as a learning practice in a mathematics classroom, one needs to study a classroom where students are

habitually engaging in proving activities. Accordingly, data from the classrooms of the purposive sample of teachers in this study is likely to be a fruitful place to look for ways in which engaging in proof can help provide opportunities for learning mathematics.

Data Collection

As part of the JAGUAR project, four multi-day lessons in the participating teachers' classrooms were video-taped over each of the two years of the study (eight lessons total). The four lessons cycles were spread out fairly evenly throughout the year (they were conducted in October, January, March, and May of each year). The teachers taught lessons around the same four tasks each year, though they made revisions to their lessons each year. Each year the teachers taught three common lessons (*The Hexagon Task*, *The Number Trick Task*, and *The Scaling Task*) and one task of their choosing. Most of the teachers taught the three common tasks first, in the order listed, and their individually chosen task last. However, two of the teachers started with their own task and then did the three common tasks in the order listed.

The camera followed both the teacher and students during whole-class discussions, and it followed the teacher during small group work. All student written work was collected after each lesson, and lesson videos were transcribed. The teachers were asked to watch videos of their lessons within 48 hours of teaching the lesson, and to write reflections about their lessons. The reflections included focused questions such as: *What is the primary goal of your lesson?*, *Did what the students do align with what you*

expected?, and *Pick one instance involving justification that you thought went well and explain what you valued about it.*

Data Reduction and Analysis

The analysis of the data corpus proceeded in phases that were consistent with grounded theory (Strauss & Corbin, 1998). In the first phase of analysis, I watched each lesson without interruption. I also examined the teachers' journals for evidence of their instructional goals for each lesson, and for evidence of how the teachers thought that proving activities promoted learning during the lesson¹¹. The point of this phase of analysis was to develop a global view of each lesson.

In the second phase of analysis, I watched the video again, looking for significant moments or critical events (Maher & Martino, 1996). In particular, these were events where students seemed to struggle with an idea related to the problem at hand and engaged in proving related to the struggle. As I documented these events, I highlighted the associated transcript excerpt and noted why I deemed each episode significant. These two phases of analysis (conducted on the entire collection of lesson cycle videos) resulted in the identification of approximately fifty *critical events*.

I went through the lesson videos task by task. That is, in order to focus my attention on the details of each task, I conducted the first two phases of analysis for each of the teachers' classrooms related to a given task before moving on to another task. For

¹¹ Although I examined the journals to gain a global perspective on the lessons, with one exception these did not point me towards anything useful for the purposes of this study. So they are not discussed in detail in this paper.

each task, I conducted both phases of analysis on data from a given teacher's class before moving on to data from another teacher's class.

In the third phase of the analysis, I looked more carefully at each of the critical events. As I did this, I examined the learning opportunities that were created by engaging in the proving activities and built explanations of how proving created those learning opportunities. I then generated a narrative describing how proving created an opportunity for learning in each episode.

In the fourth phase of analysis I began to look for commonalities between the stories that emerged. My first step in this process was to compare the role proof seemed to be playing in these episodes with the role proof is purported to play in the field of mathematics (de Villiers, 1990). As a result of this analysis, I noticed that identifying the key ideas of arguments contributed to how proofs promote learning in the classroom in several ways.

Consequently, in the fifth phase of analysis, I reanalysed all of the critical events to identify a subset that would allow me to explore the role of key ideas (of arguments) in creating learning opportunities. I looked for moments when students identified or reflected on key ideas in their proofs, and investigated what they did with the key idea once they identified or reflected on it. This analysis led me to focus on a subset of six relevant episodes. These were episodes where the essence of the proof, or the *key idea*, was explicit and where the students continued to engage in mathematical activity after the key idea was stated. During this phase of analysis, I wrote narratives describing the role key ideas seemed to play in each of the episodes.

In the final phase of analysis I categorized the ways the students interacted with the key ideas, and the learning opportunities that those created. In particular, I noticed that students generalized the key ideas, applied the key ideas, and explored the underpinnings of the key ideas. In the next section I will discuss these in more detail.

I wish to note that I do not necessarily chronicle the student learning that may have occurred in given episodes in the observed classrooms. Instead, I followed Cobb, Boufi, McClain, and Whitenack's (1997) perspective that reflective discourse "constitutes conditions for the possibility of learning, but that it does not inevitably result in each child reorganizing his or her activity" (p. 264). Accordingly, my focus will be on understanding ways in which engaging in proving can create opportunities for learning to occur, rather than documenting that learning did occur in these instances.

Results

In this section I describe three ways in which engaging in proving helped create opportunities for learning mathematical content in several middle school classrooms. In particular, as students identified and reflected on the key ideas in their proving activities they generalized the key ideas, applied the key ideas, and explored the underpinnings of the key ideas. By identifying the key idea I do not only mean identifying the key idea in an already constructed proof. I also mean coming up with the key idea in the process of creating a proof. In either case, identifying the key idea could mean recognizing or understanding the idea that is the essence of the proof or realizing that the idea is the essence of the proof. In this way, opportunities for learning occur both in the process of

creating a proof and in the process of analyzing an already created proof. Moreover, opportunities for learning occur in the process of creating or identifying the key idea, as well as after it is identified.

In the following section I elaborate on how identifying and reflecting on the key idea creates opportunities for learning. In doing so, for each of the three ways in which my analysis suggests that proving can create learning opportunities, I provide an example of a classroom episode to illustrate how proving can create the given opportunity for students to learn mathematics. In these examples, I describe the problems that the students worked on, the student responses that those problems inspired, and how proving helped create opportunities for the students in that classroom to learn mathematics.

Example 1: Generalizing a Key Idea of a Proof

As was discussed earlier, proof can lead to the discovery of mathematics because, at times, the key idea of an argument can be generalized to make a stronger statement than the one that is being proven. That is, by analyzing a proof, an individual can identify the essence of the proof, or the key idea (Raman, 2003) of the proof. Once the key idea is identified, the proof can be generalized to any object that meets the constraints of the key idea. So analysis of a proof can lead to the discovery of mathematics because it can help mathematicians generalize their results and it can motivate the need for new concepts. Consequently, one might predict that one way in which engaging in proving can create opportunities for students to learn mathematics is by helping students generalize the key idea of a proof. This is the case in the example discussed below,

proving promoted learning by creating an opening for students to extend their activity on a specific problem to think about a more general mathematical relationship.

The task. This episode occurred while students were working on *The Number Trick Task*, displayed in *Figure 3* below. On the surface, *The Number Trick Task* involves a number trick, but at the heart of the problem are several deeper mathematical issues including symbol use, generalization, and the distributive property. When using this problem, the teachers in the study had goals related to looking at important mathematical issues, rather than the surface features of the problem such as answering the simple question of whether Jessie will get the same answer when she applies these two processes to a given number. In the episode discussed below, I will illustrate how asking students to prove their answer helped a class explore one such issue: the distributive property.

Jessie discovers a cool number trick. She thinks of a number between 1 and 10, she adds 4 to the number, doubles the result, and then she writes this answer down. She goes back to the number she first thought of, she doubles it, she adds 8 to the result, and then she writes this answer down.

Here is an example:

Jessie thinks of the number. 5	Jessie goes back to her number. 5
She adds 4 to her number. $5 + 4 = 9$	She doubles her number. $5 \leftarrow 2 = 10$
She doubles the result. $9 \leftarrow 2 = 18$	She adds 8 to the result. $10 + 8 = 18$
She writes down her answer. 18	She writes down her answer. 18

Will Jessie's two answers always be equal to each other for any starting number between 1 and 10?

Explain your reasoning.

Does your explanation show that the two answers will always be equal to each other for any number (not just numbers between 1 and 10)?

Explain your answer.

Figure 3. The Number Trick Task.

In the first case Jessie chooses two numbers, adds them, and then doubles their sum. In her example, Jessie uses the number 5, so she calculates $(5 + 4) \times 2$. This same process could be expressed more generally to involve any number, as opposed to the specific number 5, as $(a + 4) \times 2$. In the second case the specific example she looks at could be represented as $(5 \times 2) + (4 \times 2)$, which can be more generally represented as $(a \times 2) + (4 \times 2)$. To prove that her claim is true involves proving that $(5 + 4) \times 2 = (5 \times 2) + (4 \times 2)$, or more generally, that $(a + 4) \times 2 = (a \times 2) + (4 \times 2)$. The first equation is a specific example of the distributive property and the second equation is a more general representation of it. So, in a class that has not yet learned the distributive property, answering this question involves justifying that relationship; in other words, justifying why the distributive property holds in the case of doubling. In most of the classes in this study, the students did not have the tools to express this idea generally. So, they talked generally while using a specific example to illustrate their claims. This is sometimes called a *generic example*, which is a specific example that manages to address the generality of a situation (Mason & Pimm, 1984).

In most of the classes that comprise this data set, when students began this problem, they did not immediately look at the problem more generally. Instead, they began by trying several numbers, and were convinced the number trick would always work on the basis of their empirical explorations. After trying several examples, students would offer justifications such as “I tried it with lots of numbers” or “It works for six. It works for seven. It works for 26...So, then it must work for all numbers.” An example of one student’s empirical exploration can be seen in *figure 4* below:

Handwritten mathematical work for 'The Number Trick Task' showing calculations for the number 6. The work is organized into two columns by a dashed horizontal line. Above the line, the number 6 is written, followed by the equation $6 + 4 = 10$. Below the line, the number 20 is written. In the right column, below 20, is the equation $6 \times 2 = 12$, followed by another 20. In the left column, below the 20, is the equation $104 + 4 = 108$, followed by two instances of the number 216 stacked vertically.

Figure 4. An example of empirical reasoning on *The Number Trick Task*.

In particular, this was the case in the seventh grade class I will discuss. At first the students tried to use empirical explorations to justify their claim. However, the teacher did not let the students make a generalization based only on empirical trials. Instead, he pushed the students to prove that the number trick would always work. When the teacher did that, the students began to talk about the numbers they were using in a general way by using a generic example. Using a generic example allowed the students to identify the key idea of the proof and begin to extend the specific idea to a more general form of the mathematical relationship.

The proof and the generalization. A student, Laura, began her proof that the number trick will always work by stating that “the second equation broke down the first equation.” When she elaborated what she meant by this, Laura talked about the number trick from Jessie’s perspective. Laura explained that, “when [Jessie] added eight, she might have imagined first the eight equals four times two, which she did in the first equation when she added five plus four and then doubled it.” Laura is arguing that Jessie thought $10 = 5 \times 2$ and $8 = 4 \times 2$, so when she adds $10 + 8$ she is actually adding $(5 \times 2) + (4 \times 2)$. Laura adds that this is the same thing Jessie did in the first equation where she computed $(5 + 4) \times 2$.

Laura then left Jessie’s perspective, and explained why, mathematically, what Jessie was purportedly thinking worked. She explained that, “to understand that, you must realize that four is still part of the equation even though it was smushed in with five. You did double four, but it was part of the five then.” Here Laura is saying that each of the individual numbers gets doubled when the sum gets doubled. In particular, she is noting that the 4 is part of the sum, so when one computes $(5 + 4) \times 2$, one is doubling the 4 and the 5. In other words, doubling that sum is the same thing as doubling the individual numbers and adding the doubled numbers together (i.e. it is the same thing as computing $(5 \times 2) + (4 \times 2)$).

In other words, in this part of her argument, Laura is breaking down the separate parts of the equation and is focusing on the fact that the sum is comprised of the addends, so doubling the sum means doubling each of the addends and adding them together. In this part of her justification she stopped talking about the specific number she used to test

the number trick, and began to talk about this relationship in a general way. She did this when she talked about the four being “smushed” in with the five; by doing this she began to talk about the sum as its own object rather than as the specific numbers that comprise it. In particular, her argument did not rely on the fact that she used the number five. That is, she could replace the number she used without changing what she said in her argument in any way.

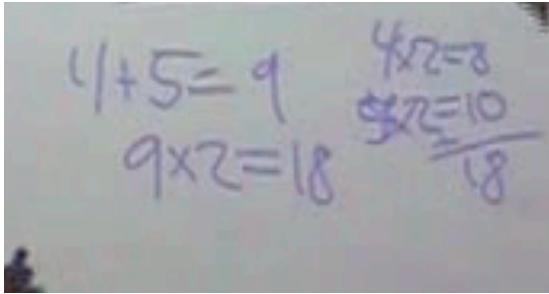
There is a substantial difference between how Laura talks about the addends, and how she talks about the number to be distributed. That is, she relies on the fact that when you multiply by two you double what you multiply. However, she does not use any such properties about the addends. This suggests that although she is not thinking about a general version of the distributive property (she is only talking about the case of doubling), her argument is talking about the addends in a general way.

As Laura continued her justification, she began to present the mathematics in an even more general way. Laura was still using the specific numbers 4 and 5, but she began to talk about the mathematics in the problem (a specific case of the distributive property, that of multiplying by 2) in a more general way. That is, she began to talk about the numbers in a way that no longer focused on the context of the *Number Trick Task*. As she talked about the mathematics in a more general way, Laura wrote a list of equations (shown in *Figure 5*) on the board that looks like a fairly typical illustration of the distributive property. She explained her reasoning:

“ So basically, when you took four plus five equals nine
[writes $4 + 5 = 9$], and then multiplied it by two you got, like,
18. But you also multiply... you could have multiplied the four

by two equals eight [writing $4 \times 2 = 8$], because you did do that.

They were just part of... together. So, like five [writes $5 \times 2 = 10$ under $4 \times 2 = 8$]... You did do that... it's like adding them together, and that's the same."



The image shows handwritten mathematical work on a light-colored surface. On the left, the equations $4 + 5 = 9$ and $9 \times 2 = 18$ are written. On the right, the equations $4 \times 2 = 8$ and $5 \times 2 = 10$ are written, with 18 written below 10 and a horizontal line drawn above it, indicating a sum.

Figure 5. Laura's written explanation of a generic example of the distributive property.

Note that in Laura's example she uses specific numbers, but elements of her justification could describe the numbers and relationships between them in a general way. In particular, when she explains that when you multiplied the nine by two, you really multiplied the four and the five by two because they made the nine, she is highlighting the key idea behind the distributive property (or at least this special case of it). That is, she was alluding to the fact that doubling a sum is the same as the sum of the doubled addends. She did not refer to any specific properties of these numbers; rather she focused on properties of sums in general. So, in a sense, she is using the numbers as placeholders.

I am not claiming that Laura necessarily realized that she was talking about the expression in a general way, or that she realized that the example she was using was not necessary to her justification. Nevertheless, from an observer's perspective it is clear that you could replace the addends in the expression without changing her justification. So it is likely that Laura is thinking generically about this example in such a way that encompasses the essence of the distributive property (in the case of doubling). However, whether or not she is thinking generically, her argument could easily be leveraged at the classroom level to get the class to talk about the distributive property in a more general way. For instance, the teacher could take Laura's argument and remove the addends to talk about how the argument would change with different numbers in place of the addends Laura used. Since Laura's argument did not use any specific properties of those numbers, her argument would not have to be modified by replacing the addends. Although Laura does not use any specific properties of the addends in her justification, she does use specific properties about the multiplier, two. Therefore Laura's argument could be extended to describe the general relationship summarized by the distributive property, but it would take more work than using it to generate a general rule for the specific case of doubling. Nevertheless, this argument could easily be extended to become a more explicit general argument, be it the specific case of doubling or the general case.

The key idea behind Laura's argument is that the sum is comprised of the addends, so doubling the sum means doubling each of the addends and taking their sum. In focusing on that key idea, Laura's justification could be seen to stop specifically referring to the *Number Trick Task*, and begin to describe the general numerical

relationship behind the distributive property (or a specific case of it). In this example, identifying and generalizing the key idea of a proof helped create opportunities for a student to learn mathematics by helping her recognize and state the important relationship summarized by the distributive property.

Example 2: Applying a Key Idea to a Different Problem

Identifying the key idea of an argument can create learning opportunities in other ways too. For instance, as I will describe below, recognizing the key idea of an argument can help students understand future problems they encounter. This can happen when the solution to the new problem is related to the key idea of the previous proof. In this section, I describe an example where a student uses the key idea from the proof of one problem in her solution to a different problem.

The task. In this example, the students are working on the *Hexagon Task*. *The Hexagon Task* is a well-known patterning problem; the exact phrasing of the problem that was used in the class discussed here is shown in *Figure 6* below.

Each figure in the pattern below is made of hexagons that measure 1 centimeter on each side.

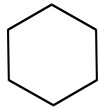


Figure 1: perimeter = 6 cm

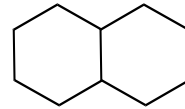


Figure 2: perimeter = 10 cm

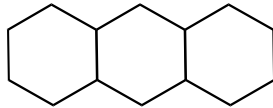


Figure 3: perimeter = 14 cm

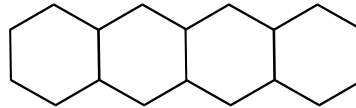


Figure 4: perimeter = 18 cm

If the pattern of adding one hexagon to each figure is continued, what will be the perimeter of the 25th figure in the pattern?

Figure 6. The Hexagon Task.

In the classes involved in the JAGUAR project, most of the students began this problem by making a table of the perimeters of trains of hexagons of different lengths.

An example of this is shown in *Figure 7*.

1	2	3	4	5	6	7	8
6	10	14	18	22	26	30	34

Figure 7. An example of a table displaying the perimeters of hexagon trains of different lengths.

As many of the students made such a table, they quickly realized there was a pattern of increasing the perimeter by four with the addition of each hexagon, and they were quick to use this pattern to generate the answer to the problem. In doing so, however, the students assumed that the pattern they observed holds, without looking for a reason the pattern might continue to hold. Below is an example of a student who used this sort of invalid empirical reasoning. The student articulated that his solution strategy was to add four each time, but could not provide a meaningful justification for the strategy of adding four each time.

Jeremy: The pattern is plus 4.

Teacher: I love that, that's great, but where does that come from?

Jeremy: Um...

Teacher: OK. So, I understand where you got the pattern from, but why does that pattern exist?

Jeremy: Uh, oh, because...

Teacher: So you're probably going to have to think about that a little bit.

The teacher in this example valued the idea of knowing why the strategy worked, so he pushed Jeremy to connect his solution strategy to the problem context, and justify why it worked. As the students were pushed to justify why adding four each time a hexagon is added to the train would lead to a correct answer, the students began to explore how the problem worked. It was this press for justification that provoked the students to explore how the problem worked, which created an opportunity for learning.

The first proof. After being pushed to explain why the pattern they observed held, the students began to explain how the pattern of adding four arose from the hexagon context and how that connected to finding an answer to the problem. There are several different ways to justify the pattern the students observed. One common way is to argue that when a hexagon is added to the train of hexagons, two side are subsumed into the inside of the train, meaning the net result of adding a hexagon is adding four sides to the perimeter. This argument is illustrated in *Figure 8* below, and described in more detail below.

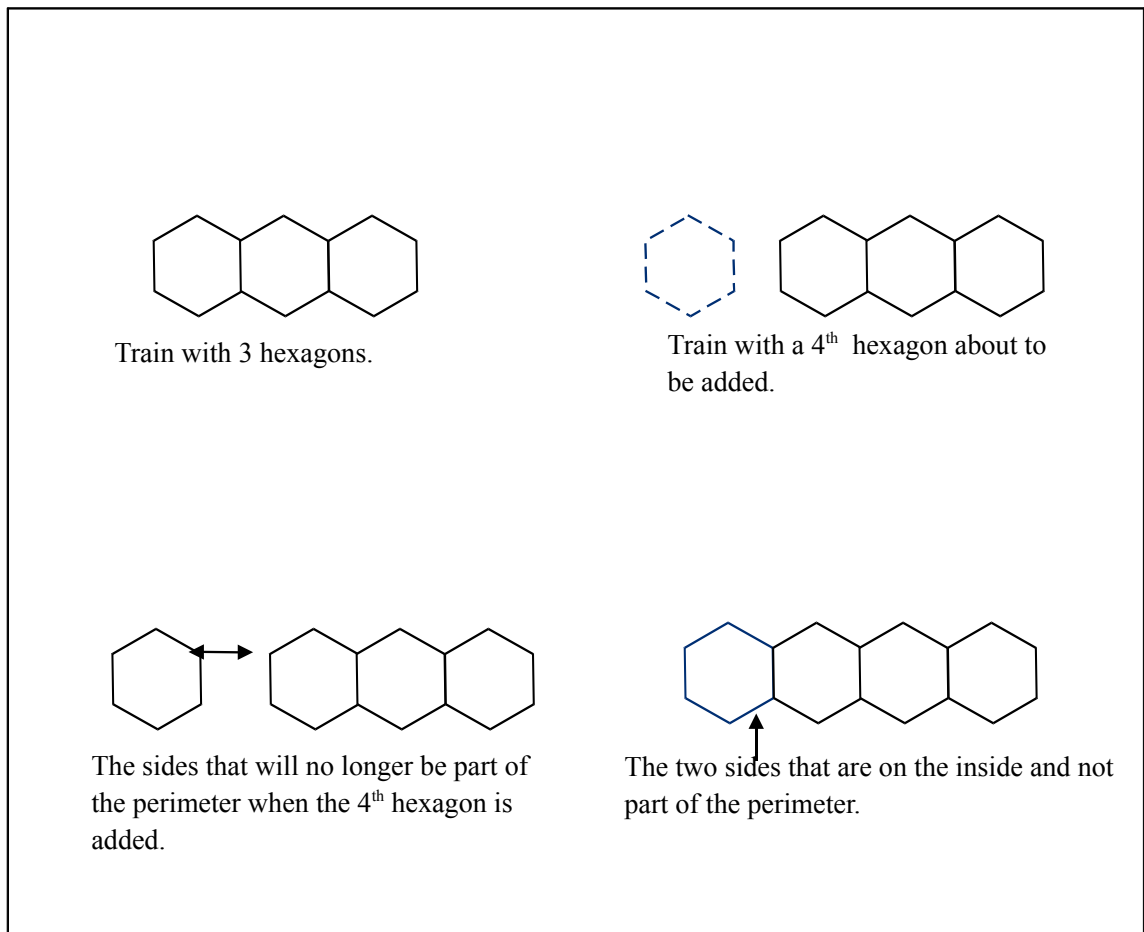


Figure 8. Illustration of one justification for adding four each time.

One student articulated this argument by stating:

“The pattern is plus four because each time you add a six-sided hexagon one side is taken off, when you add a hexagon, one side

is taken off the previous hexagon, because the sides are no longer part of the perimeter.”

In *Figure 8* the first picture shows a train with three hexagons, and the second picture shows a fourth hexagon about to be added (the dotted hexagon). Although hexagons have six sides, only five sides remain on the outside of the shape when the hexagon is added, since the sixth side overlaps a side from the original train. This is why the student says “one side is taken off each time you add a hexagon”. That is, only five sides are added to the new train rather than six. Additionally, as the student explains, “one side is taken off the previous hexagon because the [side is] no longer part of the perimeter”. The third and fourth pictures show the two sides that are on the interior of the four hexagon train, and therefore are not part of the perimeter: one side of the hexagon being added on and one side that was part of the perimeter of the three hexagon train. Thus, in a sense, adding a fourth hexagon means adding five sides and subtracting one. The net result of this change is adding four sides.

The key idea of this proof is that when you put two hexagons together two sides do not contribute to the perimeter; one side from each hexagon is in the interior of the new figure. Consequently, for this proof, that means that each time another hexagon is added to the train, the perimeter increases by four. In the example below, I will show how a different student applied this key idea to a proof addressing a different problem later in the class period.

The second proof. While working on the *Hexagon Problem* (as shown in *Figure 5* above), not all students used the solution described above. Another strategy many students used was to try to apply proportional reasoning. For example, some of the students found the perimeter of a train of five hexagons, and then multiplied that value by five to determine the perimeter of a train of 25 hexagons (see *Figure 9* below).

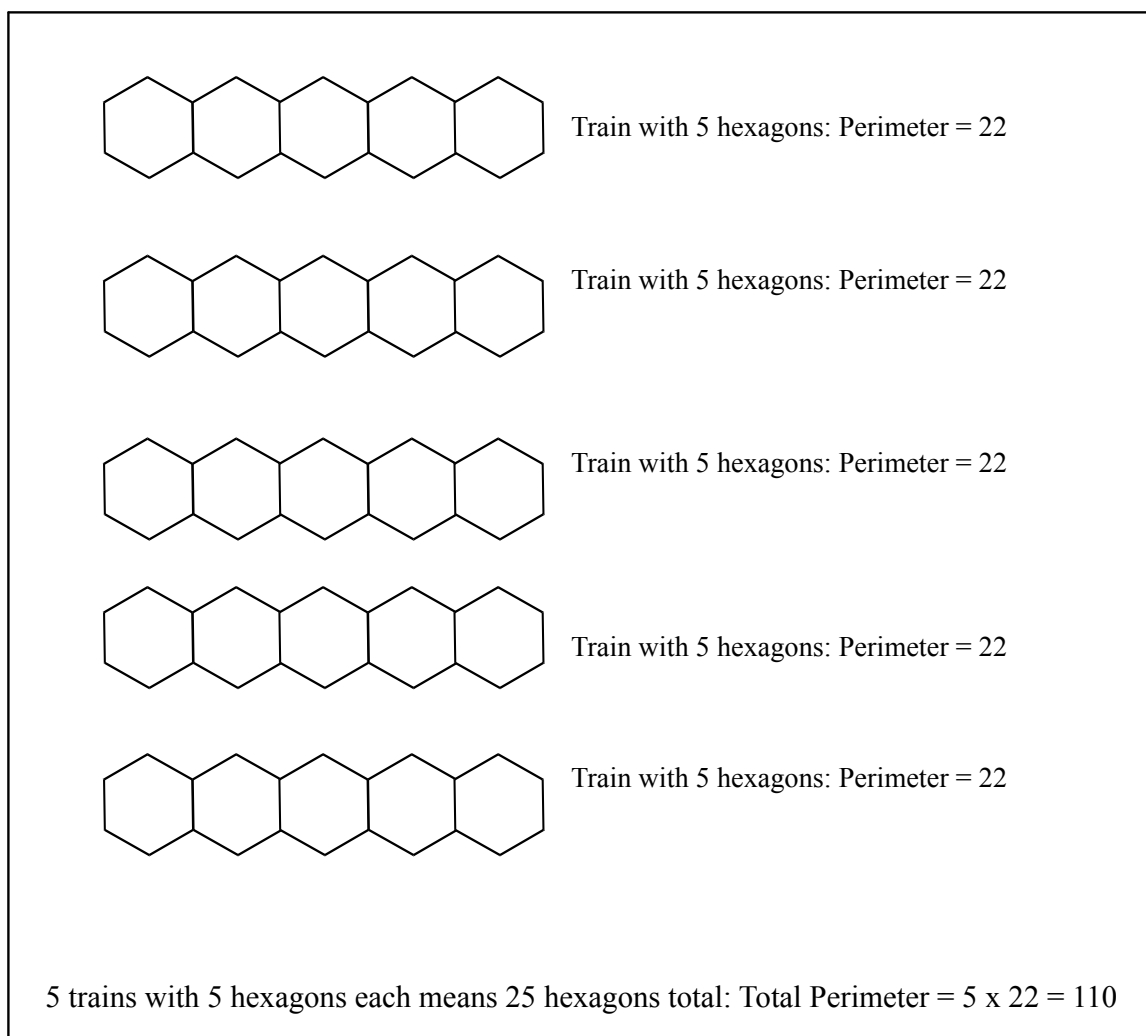


Figure 9. Proportional reasoning solution to The Hexagon Task.

These students using this method argued that this method worked since there were five groups of five in twenty-five. However, students using the proportional reasoning method did not arrive at the correct perimeter.

This strategy was met by a variety of responses from students and teachers in the various classes. In the class in this example, many of the students immediately responded to this strategy by stating that the answer determined in this way was incorrect, and then described their own method of repeatedly adding four to find the answer. This argument was meant as a means of justifying that the answer found by multiplying was wrong. That is, the students tried to argue that if the two answers were different and their answer was right, then the other answer must be wrong. It is true that this would justify that the answer was wrong, but it would not confirm if the strategy was wrong or if the student merely made a computational error. Moreover, the teacher wanted the students to know more than that the strategy was wrong, he wanted them to know why it was wrong. So the teacher in this class pushed for more, he said “that still doesn’t help us with what’s, what could be wrong with [this solution].” In doing that, he pushed the students to prove why the proportional reasoning method was in fact wrong.

Eventually Adam, a student in the class, was able to justify why the multiplicative strategy did not work. Adam argued:

“But what happens is when you multiply the 5, 5 times 5 does equal 25, but when you multiply 22 by 5 you get 110, but what happens cause when you put the 25 together it’s going to be like one long strip. So the parts where they, where each five comes together, you’re going to have to take those away, ‘cause they’re part, they’re inside the shape, not the perimeter.

It means you have to take 2 away for each, um, line, and then you have to take 2, 4, 6, 8, you have to take 8 away from the 110. So you could have done that and taken away the 8 and you would have still gotten the right answer.”

Adam’s argument was that it is true that five groups of five make twenty-five, but that when a train of 25 hexagons is made the five 5-trains are pushed together. Adam thus understood that some of the edges that are part of the perimeters of the five 5-trains are on the inside of the 25-train, and so there are parts of the perimeters of the five 5-trains that are not part of the perimeter of the 25-train (see *Figure 10*).

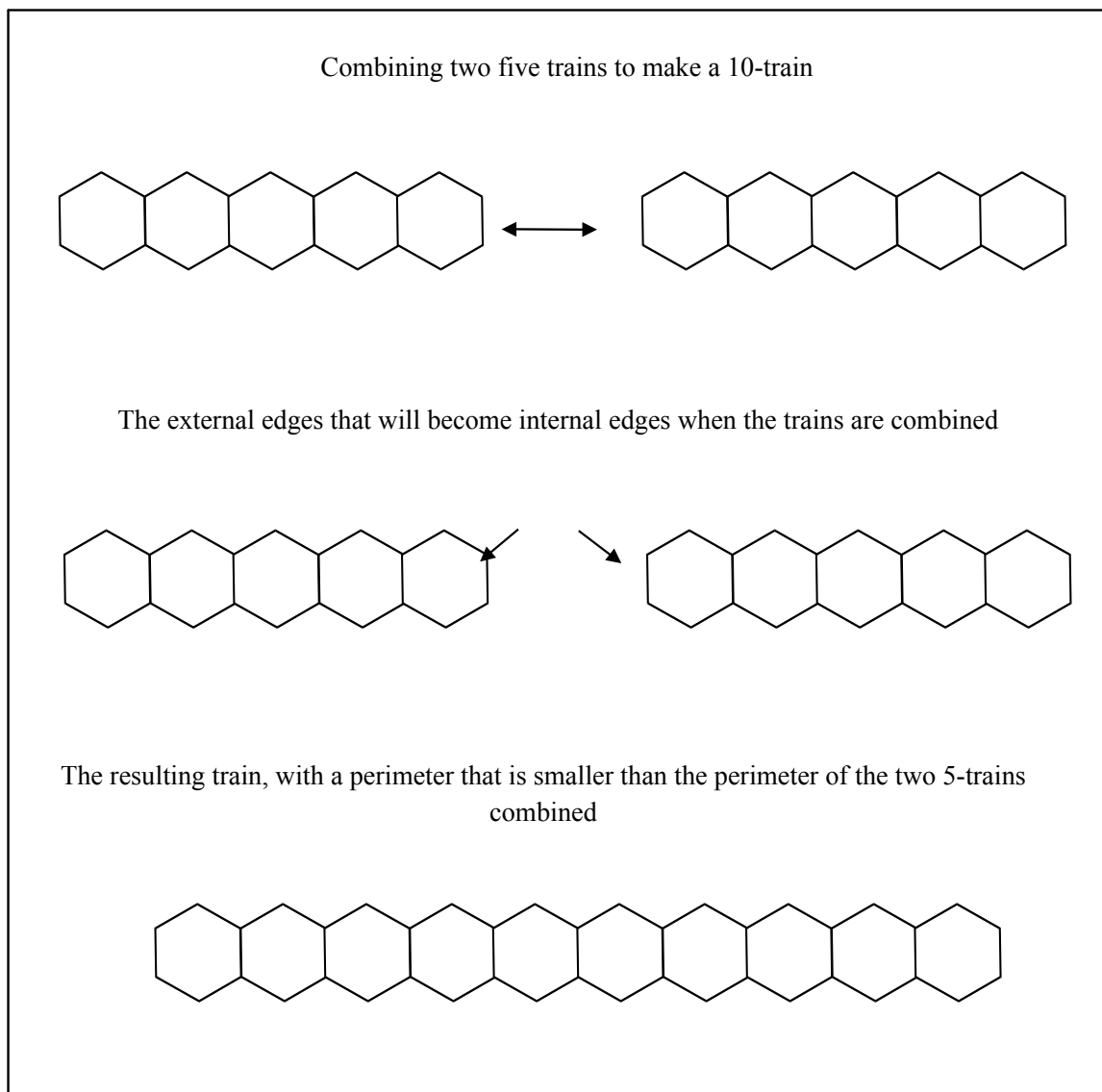


Figure 10. Combining two 5-trains to make a 10-train.

Adam understood that the proportional reasoning strategy leads to over-counting.

Therefore, to find the perimeter of a 25-train, a person needs to subtract off the inside edges (see *Figure 11*). As Adam explained, in this case, that means subtracting off two

sides for each train that is added to the original train. So that means subtracting off eight sides.

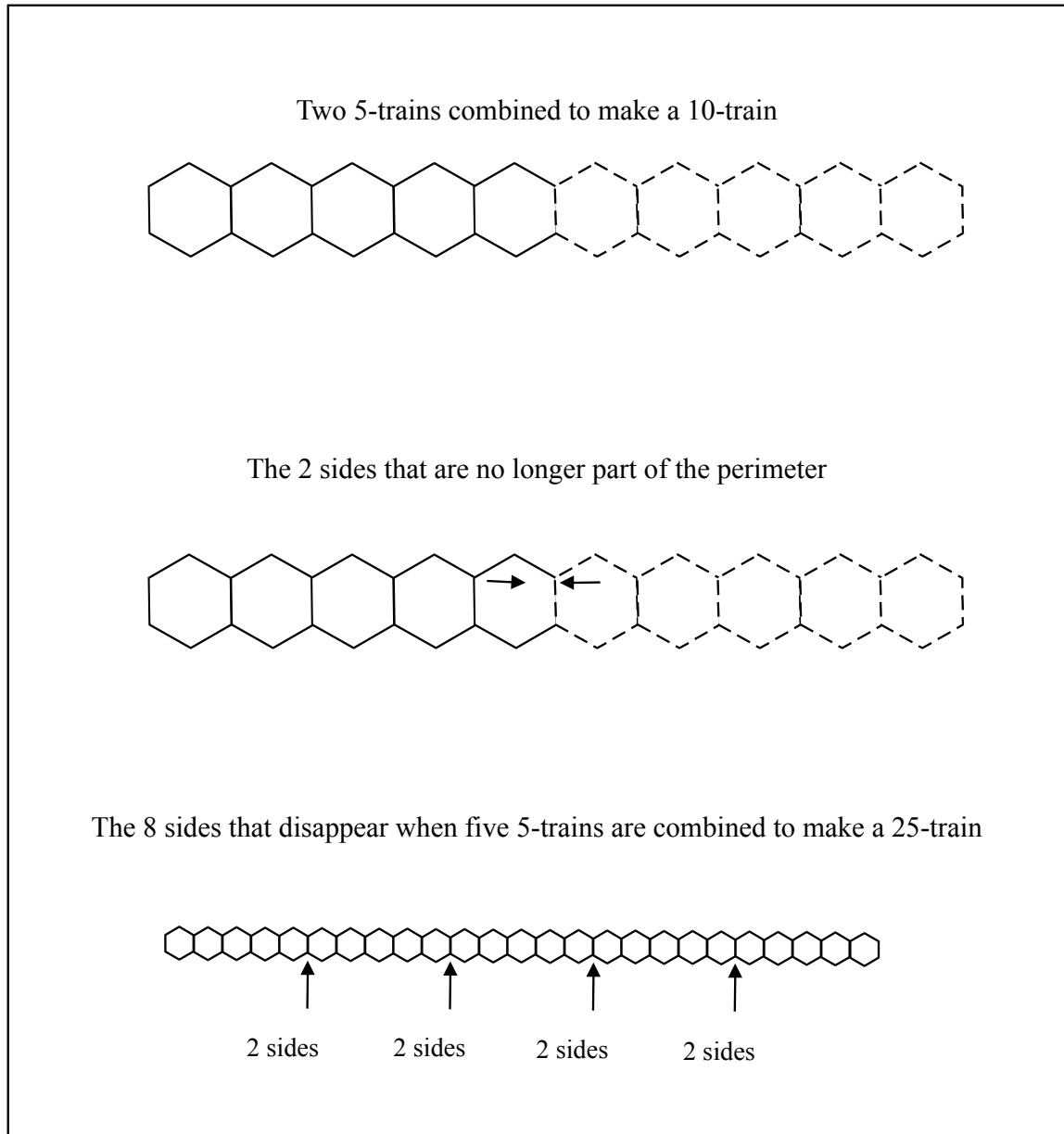


Figure 11. Modifying the proportional reasoning strategy.

This means that one can modify the proportional reasoning strategy by subtracting after multiplying. In other words, one can find the perimeter of a train of 25 hexagons by

calculating $(5 \times 22) - (2 \times 4)$. The 5×22 calculates the perimeter of the five 5-trains together, the 2×4 represents the sides that become internal when the trains are combined to form the 25-train, and the latter part of the expression is subtracted from the former to compensate for the over counting.

The argument Adam presented for removing these sides was based on the key idea behind the argument presented earlier in class, that when you put two hexagons together two sides do not contribute to the perimeter. Adam was able to address why the proportional reasoning strategy did not work by applying the key idea from the previous proof to this new problem. This allowed Adam to show that the answer found by using proportional reasoning was incorrect, and enabled him to develop a work-around for the limitations of the proportional reasoning strategy. In particular, it created an opportunity for him to learn that not all patterns increase in a proportional manner. This function is linear, and increases at a constant additive rate and is not a simple multiple of the figure number. That is, it increases by four every time. However, the perimeter of the first train is not four, it is six. This means that rather than just multiplying the number of hexagons by four, a student needs to take into account the perimeter of the first train in the sequence. Therefore, this discussion created an opportunity for Adam (and possibly other students) to think about ideas that lay groundwork related to the role of constants in the linear expressions and to reevaluate his ideas about solving problems proportionally. In this way, engaging in proving created several learning opportunities for students because identifying the key idea of a proof potentially gave students a new tool for thinking about other problems they may encounter.

Example 3: Exploring the Underpinnings of a Key Idea

Another way in which proving can help create opportunities for learning mathematics is by motivating students to focus on why a claim is true. Identifying the key idea behind a proof can lead to an exploration of why that idea is true. In particular, a person might examine the evidence someone used to justify their claim, or they might explore how the evidence leads to the given claim. In this way, engaging in proving can create a need for students to look at more than just how to justify their answers. It can create a need to look at why their proof (or someone else's proof) works in a given situation. Consequently, proving can be seen as an important process for promoting learning in the mathematics classroom because it can inspire a student to search for understanding.

The task. In this example the students were exploring *The Odd and Even Game* (shown in *figure 12* below). They were asked if the game is fair and to justify their answer.

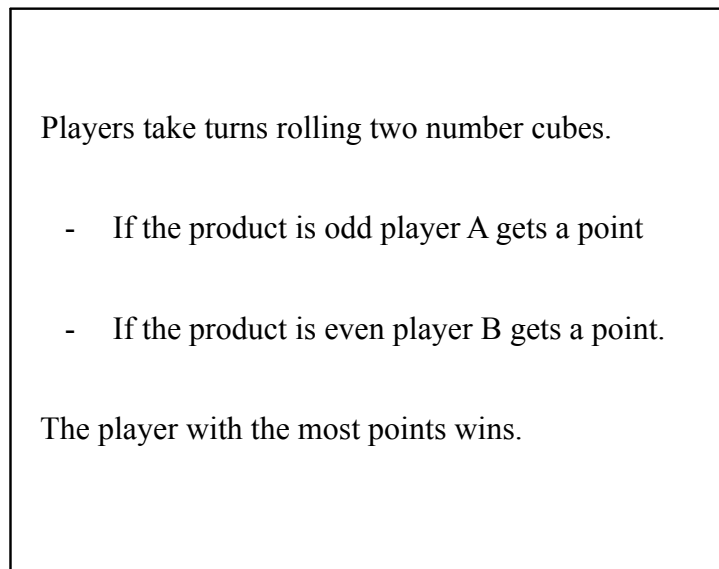


Figure 12. The Odd and Even Game.

Of course, this game is not fair because the two outcomes that can occur, getting an odd or even product, are not equally likely. Out of the 36 possible outcomes, 9 of them are odd and 27 of them are even.

Although the game is not fair, at first, most of the students believed the game was fair. The most commonly cited reason was that there were only two choices, so you had an equal chance of either outcome. For instance, one student said “you’re either going to get even or odd...[so] each have a 50% chance.” Some students also argued that the game was fair because the point distribution was fair. For example, one student said “because [they] also each [get] a point when they win.”

The proof. Some students did not believe the game was fair. In particular, one student argued that the game was unfair because the two outcomes were not equally likely.

Melanie: It's not fair because 27 out of 36 times the products are going to be even.

Teacher: What do you mean?

Melanie: Well, when you multiply all of the numbers, like when you do the theoretical data... the number of even products is 27 out of 36.

Melanie justified her answer empirically, by showing all of the possible outcomes and counting which ones were odd and which ones were even. By doing this, she was able to prove that the two outcomes were not equally likely. Another student, Bill, responded to Melanie's argument by adding an explanation of why the outcomes were not equally likely. He argued that:

“What a lot of people didn't realize is that when you multiply two even numbers or an odd and an even number the outcome is always even, but if you multiply an odd by an odd ... you get [an odd].”

The key idea behind both of these arguments is that the two outcomes are not equally likely. The two students cited different evidence to back up their claim that the outcomes are not equally likely, but both of their arguments relied on the same fundamental reason for arguing the game was unfair.

The investigation. The justifications the students offered as to why the game is not fair could lead to many different discussions. For instance, the students needed to discuss what it meant for a game to be fair. In principle, they would need to agree that the fact that the two outcomes were not equally likely was an important criterion to decide if the game was fair. That fact was already established in this class. So, the students did not need to discuss that topic. In fact, since that fact was already established, the students realized it was the key idea of any argument addressing the fairness of a game. Although they realized they were discussing the key idea of the argument, they did not agree on the veracity of the statement. That is, they did not agree on the likelihood of each of the outcomes. Therefore, they needed to determine if the two outcomes were in fact equally likely.

The core of an argument can be seen as having three components (Toulmin, 1969): the *data*, the *warrant*, and the *backing*. When justifying a claim, a person provides evidence (or *data*) that presumably leads to that claim. The *warrant* explains how the evidence leads to the given claim and the *backing* explains why the warrant is true. In this case, the *warrant* is that the two outcomes are not equally likely, and the *backings* are the various arguments presented for why the two outcomes are not equally likely. Therefore, in this case, the students agreed on the warrant, but they needed to discuss the backings the students offered for their arguments.

As a result, the class had a lengthy discussion about the empirical reasoning strategy. Students who had used that strategy did not all agree on the numbers Bill presented. Some students had not made organized lists, and had missed out on some of the answers. Other students had crossed out some answers they thought were double

counted (i.e. thinking a 3 on the first die and a 1 on the second is the same as a 1 on the first die and a 3 on the second). A few students had made simple computational errors. These discussions allowed students to see the importance of organized lists, to catch simple arithmetic mistakes, and to realize that similar outcomes need to be counted as distinct in this situation. In this way, when the students recognized that the key part of the argument was that the outcomes were not equally likely, they became motivated to investigate the backing of that argument. This whole process provided several opportunities for learning.

In the course of constructing a proof, one of the students asserted a general principle of mathematics: that when you multiply an even number by another number the outcome is always even, but if you multiply an odd number by an odd number you get an odd number. In this particular class, there was not a follow-up discussion to elaborate on this important conjecture—likely because of time constraints or because it would have distracted the class from the topic at hand. Nevertheless, the lesson could be designed differently to incorporate a full discussion of that claim as well.

Regardless of which backing was discussed, identifying the key idea of a proof created several opportunities for exploring important ideas. This is because reflecting on the key idea made the students aware of the need to question the idea, and, in turn, to explore the backing given in the proof. In this way, engaging in proving created an opportunity for learning by creating a motivation for students to explore the underpinnings of the key idea and a context for doing so.

Discussion

In the examples discussed above, proving created opportunities for learning mathematics. These opportunities are tightly connected to other research in mathematics education. For instance, the ways in which proof created these learning opportunities are similar to ideas discussed in previous research on proof. Additionally, the learning opportunities described in this article relate to theories of learning discussed in other research on the learning of mathematics.

Research on Proof. In the first example, *The Number Trick* example, proving created an opportunity for learning mathematics because the student eventually described the key idea of her proof in a way that could be seen as a generalization of the results of the proof, leading to the description of a specific case of the distributive property. In this example, we can see proof functioning in the classroom in a way that is similar to how it functions in the field of mathematics. In particular, one of the ways proof serves as a means of discovering mathematics that de Villiers (1990) describes is that identifying the key idea of a proof can lead to a generalization of the idea being proven. This is much like what happened in this classroom example. In this example, the key idea of the justification that the number trick worked provided an opportunity for generalization of the idea in order to develop the distributive property. Therefore, one can see that proof can play this role in the classroom as well as in the field of mathematics.

In the second example, *The Hexagon Task*, engaging in proving helped a student gain a new tool for solving problems. An interesting part of this example was that the student applied the key idea of a proof to a different problem. That is, the student

abstracted the key idea of the proof to generate a strategy for solving a different problem. In particular, in this example, the key idea of the justification of a procedure for calculating the perimeter in the hexagon problem became the basis of a justification for a different procedure. This is also similar to the role of proof in research mathematics. For instance, Weber (2010) explains that mathematicians often look to proofs to learn new methods or strategies that are applicable to different contexts. In the same article, Weber described an example of children using a method that was used to solve one problem (that of using a binary representation) to solve other problems. From these examples we can see that students of mathematics at all levels can look to proofs to discover both new methods and new strategies for solving problems just as mathematicians do.

In the third example, *The Odd and Even Game*, proving created a learning opportunity because it provoked the students to explore if and why a statement was true. Engaging in proving provoked the students to search for why things are true, which enabled them to gain enlightenment with respect to mathematical ideas. Similarly, mathematicians look to proofs for insight into ideas (de Villiers 1990). Kidron and Dreyfus (2010) also noted a similar idea in their paper about an individual student studying bifurcation points. In their study, much like is commonplace in research mathematics, the enlightenment the student gained was directly related to the task the student was exploring, whereas in the *Odd and Even* example the enlightenment was related to mathematical ideas behind the task rather than to the task itself. In these ways, we can see that proving can help students develop an understanding of mathematical ideas in several ways.

Research on Learning. The learning opportunities discussed in this paper are also consistent with ideas discussed in previous research on the learning of mathematics. For instance, in the first example, proving created an opportunity for learning mathematics because the key idea of the proof of the *Number Trick Task* led to statements that could be extended to a generalization of the results of the proof. This is resonant of the *Realistic Mathematics Education* (RME) construct of *transformational record* (Rasmussen & Marrongelle, 2006). *Transformational records* are a way teachers can advance the mathematical agenda by extending student thinking. In particular, they are artifacts, such as graphs, equations, or verbal statements, that a teacher uses to move students' thinking forward. The argument Laura presented in the context of *The Number Trick Task* could be used as a *transformational record* to help students articulate the general relationship represented in the distributive property (or at least the specific case of doubling). To do this, the teacher could write the argument on the board and underline the addends Laura is using. The teacher could then ask if the numbers could be replaced by other numbers and have the number trick still work. Since Laura's argument does not rely on any properties of the specific addends in the problem, it would be easy to use her argument to justify why the number trick would work with other numbers. In this way, the teacher could build on the students' mathematical reasoning and push them to think in a more general way. Although Laura does not use any specific properties of the addends in her justification, she does use specific properties about the multiplier, two. Therefore using Laura's argument as a *transformational record* to describe the general relationship summarized by the distributive property would take more work than using it to generate the specific case of doubling. Nevertheless, Laura's argument could be used as a tool to

create an opportunity for students to learn about the distributive property, be it the specific case of doubling or the general case.

It was the activity of proving why the number trick worked and the act of communicating that proof to others that created this opportunity to learn about the distributive property. Asking students to engage in proving pushed Laura to articulate an argument that uses numbers in what could be seen as a generic way. It was in justifying her answer that Laura articulated the key idea behind the number trick, that of “smushing” the addends together. By continuing this argument, the class could extend Laura’s argument into a general one. Therefore, from the RME perspective, in this example, proving could be seen to create an opportunity for learning to occur.

In the second example, the one dealing with the *Hexagon Task*, proof created an opportunity for learning as a student applied the key idea from a previous proof to the solution of a new problem. This is like the construct that Stephan and Rasmussen (2002) discuss in their paper on math practices. In that paper, Stephan and Rasmussen study learning at the classroom level by exploring the ways in which students’ ways of reasoning became taken-as-shared within the classroom community. They used Toulmin’s model of argumentation (Toulmin, 1969) to explore when reasoning became taken-as-shared. As was previously described, this model describes how evidence is used to support a claim. The first part is the data used to justify the claim. The second part is the warrant, which explains how the data leads to the claim. The third part is the backing, which involves justifying why the warrant supports the claim. When using this model to explore when reasoning had become taken-as-shared, the researchers studied when ideas shifted roles within an argument. For instance, they looked at when ideas shifted from

being warrants that needed backing to warrants that were accepted when offered. When ideas shifted roles within an argument, they said ideas had become taken-as-shared, which they said meant learning had occurred.

This is precisely what happened in the second example. That is, in the first proof of the hexagon problem discussed above students needed to justify that two sides are absorbed into the inside of the shape when a new hexagon is added. In the second proof a student used the fact that two sides are absorbed inside the shape when a hexagon is added to the train to justify why the multiplicative strategy does not yield the correct answer. The key idea shifted roles in the argument from the first proof to the second proof. Since the students had been convinced of the soundness of the idea from earlier arguments, it was not challenged when it was introduced in the new proof. In this way, from Stephan and Rasmussen's (2002) perspective, learning occurred in this example, and the act of proving facilitated that learning.

The learning opportunities discussed in the third example also relate to the research literature on learning. In this example proving creates an opportunity for learning because students find a need to explore the underpinnings of the key idea behind their solution to the *Odd and Even* problem. Harel (2001) argues that for students to learn they must see the need of what a teacher intends to teach them. He elaborates that by need he means intellectual need and not social or economic need. In the *Odd and Even Game* example, proving that the game is not fair created an intellectual need to know which products are even and which are odd. In this example, proof creates the kind of intellectual need that Harel argues is key to supporting student learning. Consequently,

this example shows how, according to Harel's theory, proof can be seen to create opportunities for learning to occur in the classroom.

Conclusion

In this paper I explored ways in which proof can help students learn mathematics by looking at middle school students engaging in proving activities in their mathematics classes. To that end I discussed examples of learning opportunities that can be created by engaging in proof. I also discussed how engaging in proving leads to these learning opportunities and connected those opportunities to previous research on the learning of mathematics.

The three examples discussed in this paper are meant to show examples of ways in which proof can promote learning in the classroom. In particular, they are meant to show how identifying and reflecting on key ideas of proofs contribute to this process. It is likely that this is not a complete list of the ways that reflecting on the key idea of a proof can create learning opportunities, as analysis of other classrooms and other tasks would likely generate more ways that reflecting on key ideas of proofs create opportunities for learning. It is, however, intended to serve as a starting point illustrating ways proving can promote learning in the mathematics classroom. In addition, focusing on key ideas is likely not the only way engaging in proving creates learning opportunities in the mathematics classroom. Consequently, further research needs to be done to further study the interrelationship between proving and learning.

Additionally, in this paper I document the existence of several potential opportunities for student learning that occurred in several classroom episodes in order to paint a picture of the potential of proof as a learning practice in middle grades classrooms. I do not intend this analysis to be adequate to show that learning did occur for any or all of the students in the class. However, the learning opportunities discussed in this paper are consistent with theories of how students learn mathematics that are discussed in existing mathematics education research literature. So it is likely that proving can lead to learning in the mathematics classroom by creating opportunities such as those described in this paper.

It is also worth mentioning that many factors lead to the learning opportunities described in this paper. The problems used were an important factor, but just using these problems does not guarantee that students will engage in meaningful proving opportunities that create opportunities for learning. The approaches the students used, the cultures related to proof and proving established in the classrooms, and especially the teachers' actions and reactions all contributed to the opportunities for learning that arose in these classrooms. Exploring how specific details such as these supported the engagement in proving activities that created the learning opportunities that were observed was beyond the scope of this paper, but would be a valuable avenue for future research.

As was mentioned in the beginning of this article Lampert (1990) argued that students can engage in activities such as proving, but “the problem of defining what knowledge they have acquired remains” (p. 59). Although I do not define precisely what knowledge the students have acquired, this paper is an important step towards that goal

because it describes situations that create the potential for proof to promote learning in the mathematics classroom. In particular, I showed how proving creates opportunities for students to learn mathematical content related to the key ideas in their proofs. Even though many questions still remain about how proving relates to learning, this research is a helpful step for future research as it moves us toward better understanding the relationship between proof and the learning of mathematics.

Chapter 4

An Implications for Practice paper

Proof as a Learning Process

Each figure in the pattern below is made of hexagons that measure 1 centimeter on each side.

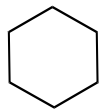


Figure 1: perimeter = 6 cm

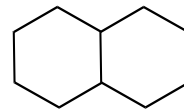


Figure 2: perimeter = 10 cm

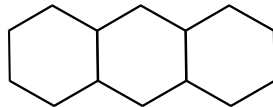


Figure 3: perimeter = 14 cm

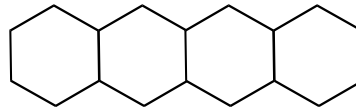


Figure 4: perimeter = 18 cm

If the pattern of adding one hexagon to each figure is continued, what will be the perimeter of the 25th figure in the pattern?

Figure 13. The Hexagon Task.

Students in a seventh grade class are in the middle of sharing their answers to the *Hexagon Task* (shown above in *Figure 13*), when a student shares an answer that her classmates disagree with. Up until that point, the students had been sharing different

justifications for solution strategies that related to the idea of repeatedly adding four to find the perimeter. Some of these justifications were based on counting the perimeter of trains of different lengths (*Figure 14, Example 1*), and some referenced the act of building the trains (*Figure 14, Example 2*).

Example 1: Some students begin this problem by counting sides to calculate perimeters of different length trains. These calculations are often recorded in an organized list or table. Sometimes students continue counting to determine the perimeter all the way to 25 hexagons. Other times students notice the pattern of increasing by four and use that pattern to calculate the perimeter of a train of 25 hexagons.

1	2	3	4	5	6	7	8
6	10	14	18	22	26	30	34

Example 2: When students notice the pattern of adding four each time, they often turn to the shape itself to look for justifications for the pattern. For instance, one student said “The pattern is plus four because each time you add a six-sided hexagon one side is taken off, when you add a hexagon, one side is taken off the previous hexagon, because the sides are no longer part of the perimeter” (see image below for an illustration of this justification).

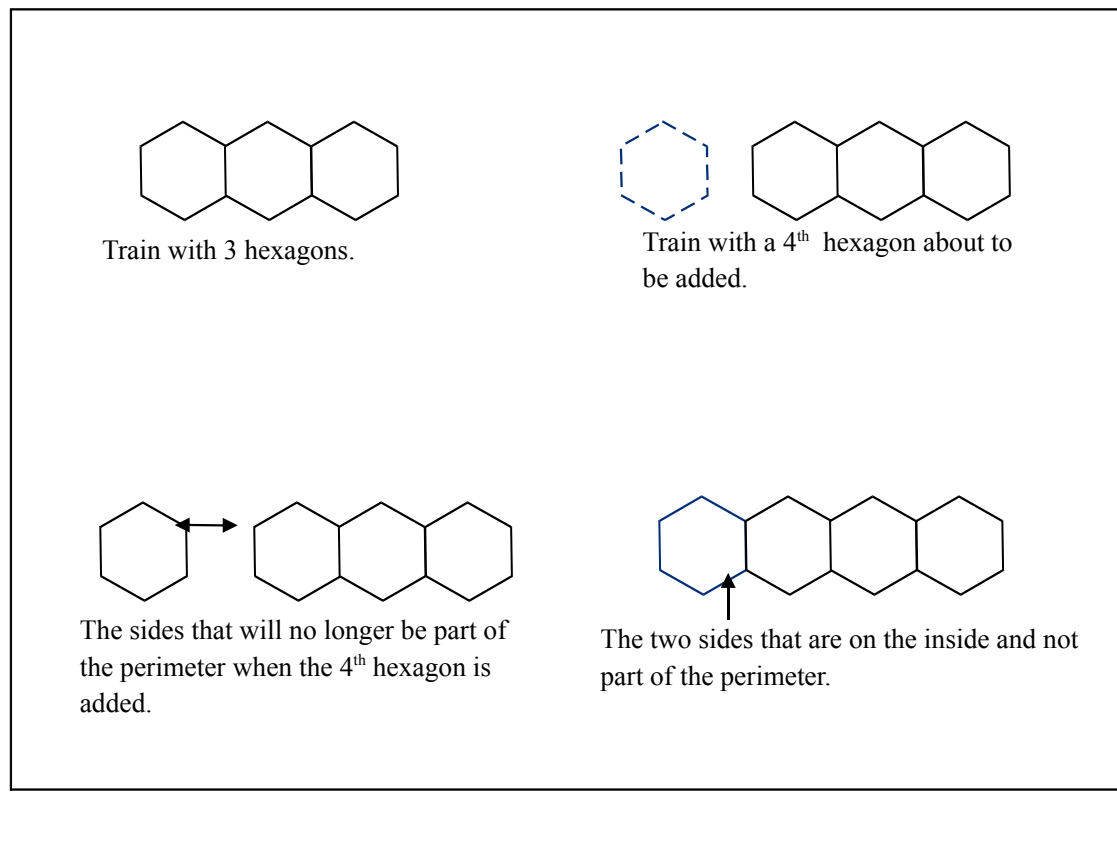


Figure 14. Examples of student justifications for *The Hexagon Task*.

Then a student shared a strategy that used very different reasoning; he used the perimeter of a smaller train of hexagons to get the perimeter of a train of 25 hexagons. This

solution strategy not only used a very different type of reasoning than the previous strategies, it also yielded a different perimeter (110 instead of 102). Consequently, the students disagreed with this solution strategy. The controversial strategy the student shared was:

“since this is figure four, I added on another one [to get a fifth hexagon]. And then I got a different answer [22] and then I multiplied it by 5 because 5 times 5 is 25 and then [22] times 5 would be the amount and I got, 110.”

As the student explained above, since twenty-five is five groups of five, the student calculated the perimeter of a train of five hexagons and multiplied that perimeter by five to find the perimeter of a train of twenty-five hexagons (see *Figure 15* below for an illustration of this method).

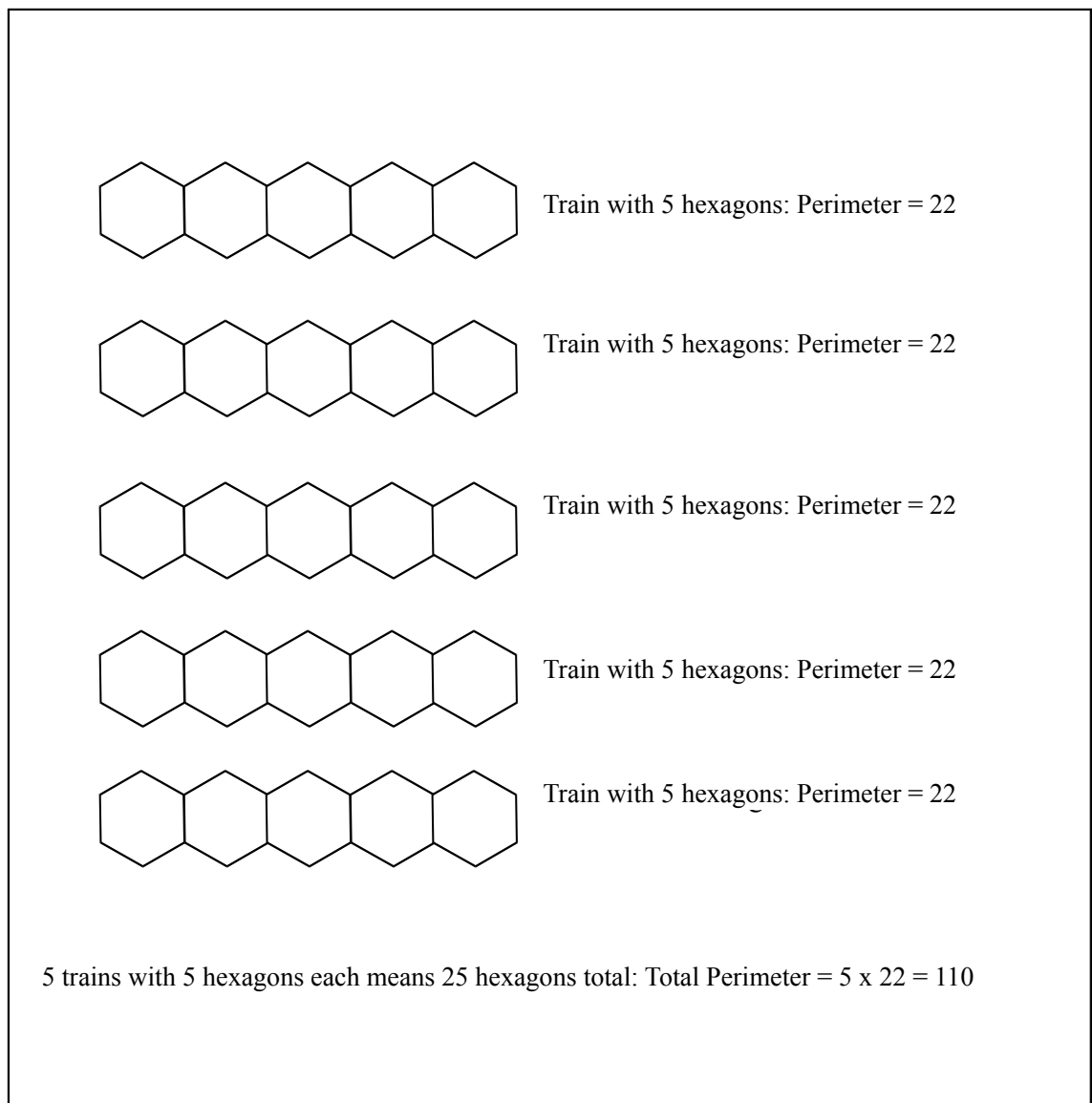


Figure 15. Proportional reasoning solution to the *Hexagon Task*.

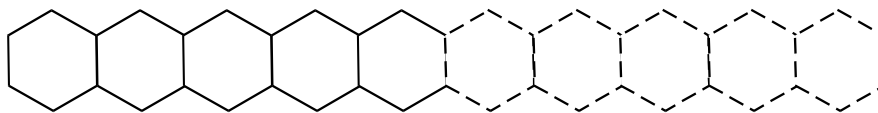
This is a creative solution, but it is slightly flawed as it counts some of the sides that end up on the inside of the figure when the full figure is formed. Since this strategy was very different from what many of the other students had done, and resulted in a

different answer, it was immediately challenged by students in the class. For instance, one student quickly said, “I say that, um, her answer is wrong.” When the teacher asked the students to justify their belief that this strategy did not work, the students argued that the strategy resulted in the wrong answer and tried to show their solution strategies that lead to the right answer. The teacher responded to the students by saying that they were sharing correct strategies to solve the problem, but that there are other ways to solve the problem as well. Then he pushed the students to prove their claims that the controversial solution strategy did not work by saying, “that still doesn’t help us with what’s, what could be wrong with [this solution].” When the teacher pushed the students to think about what was wrong with the strategy, a student was able to explain *why* the strategy led to the wrong answer. The student explained that:

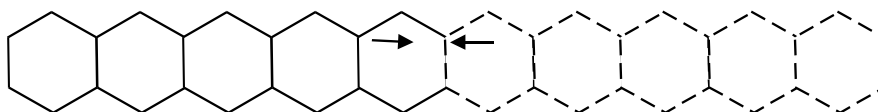
“five times five does equal twenty-five, but when you ...put the twenty-five together it’s going to be like one long strip. So the parts where they, where each five comes together... they’re inside the shape, not the perimeter.”

Figure 16 (below) shows an illustration of the sides that are parts of the perimeters of the smaller trains of five hexagons, but are not part of the perimeter of the longer train of hexagons.

Two 5-trains combined to make a 10-train



The 2 sides that are no longer part of the perimeter



The 8 sides that disappear when five 5-trains are combined to make a 25-train

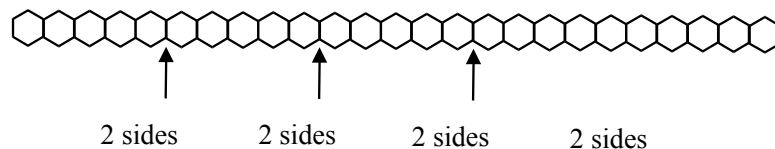


Figure 16. Combining trains of 5 hexagons to make a train of 25 hexagons.

The student adds that:

“ at each line they connect together and what you’re going to have to do is you’re going to have to take the two away, because that’s now becoming inside the shape... It means you have to take 2 away for each, um, line, and then you have to take 2, 4, 6, 8, you have to take 8

away from the 110... So you could have done that and taken away the 8 and you would have still gotten the right answer.”

So, in the end, the student argued that you can use the perimeter of a shorter train to find the perimeter of the longer train, you just need to account for the parts of the perimeters of the shorter trains that are absorbed into the inside of the longer train.

Proof as a Learning Process

Principles and Standards for School Mathematics states that proof is a “powerful [way] of developing and expressing insights” (p. 56). This view of proof focuses on proof as a way to learn other mathematical content, rather than only as content to learn. The example above illustrates this role of proof in the middle grades. In this example proving created learning opportunities by helping students develop and express their insights about the problem.

At first, many of the students wanted to dismiss the proportional reasoning strategy because it yielded a different answer than they had computed. While it is true that this could be a way to prove the answer found from the proportional reasoning strategy was wrong, it does not immediately prove that the proportional reasoning strategy itself is wrong. That is, a wrong answer can occur when a student makes a computational error with a correct method or when a student uses a flawed strategy. Pushing the students to find out what was wrong with the proportional reasoning strategy helped the students determine the strategy was flawed.

We have seen many students in different classes use this type of reasoning. In this class, the public debate about the strategy created an opportunity for the students who used this type of strategy to reflect on the flaw in their reasoning. The opportunity for reflection created an opportunity for the students to learn *why* the answer was wrong and not just *that* it was wrong. In turn, engaging in proving created an opportunity for the students to modify, rather than abandon, their original method to find the correct answer. So proving could help students using this type of reasoning revise their misconceptions and let them harness their own thinking to find an answer.

In addition to creating an opportunity for students using this strategy to revise their incorrect strategy and develop a correct strategy, engaging in proving also created an opportunity for the students to learn about proportional reasoning. In particular, it created an opportunity for the students to learn that not all patterns can be thought of as a direct proportion. This function is linear; it increases at a constant additive rate of four each time a hexagon is added, but the perimeter of the first train is not four, it is six. This means that rather than just multiplying the number of hexagons by four, a student needs to take into account the perimeter of the first train in the sequence. In this way, this episode provided an opportunity for students to compare proportional relationships to linear relationships, and to learn about the different ways in which functions can grow.

As was discussed above, engaging in proving created opportunities for learning because it pushed the students to figure out what was wrong with the proportional reasoning strategy, which enabled them to modify the strategy so that it did work. However, that was not the only learning opportunity to be found. The vignette also provides an example of a student using the key idea of one justification to discover and

correct the flaw in a completely different justification approach. In one of the proofs presented at the beginning of the task discussion, the key idea for why the perimeter grows by four each time a hexagon is added was based on the idea that when two hexagons are connected, one side from each hexagon gets lost on the inside of the train (see *Figure 14, Example 2* above). This same insight was used to explain why the multiplication strategy did not work and to figure out how to modify it. That is, the justification the student offered about what was wrong with the proportional reasoning strategy was based on an adaptation of this idea. In particular, the key idea of this later proof was that when two trains of hexagons are connected, one side from each train of hexagons gets lost on the inside of the shape. In this way, engaging in proving and identifying the key idea of a proof helped a student develop a new strategy for solving a new problem he encountered. Consequently, engaging in proving can be seen to create different types of learning opportunities in the classroom including helping students revise incorrect solutions and helping students learn new strategies for solving problems.

The Role of the Task

The task used in this class also contributed to the learning opportunities that were created. One important aspect of the task was that it involved complex reasoning, which motivated discussion around the task. Additionally, the fact that students could use a variety of different strategies to begin working on the task and that they could use a variety of different types of reasoning to justify their strategies helped create the learning opportunities discussed above. Tasks that can be solved in multiple ways give students

access to the task because students can approach them in their own way. This flexibility motivates and interest students to engage with the problem.

The fact that this task evokes multiple solutions to the problem has other benefits too. In particular, it creates opportunities for students to share, explore, and discuss alternative ideas. This task and its implementation created a situation where students had different justifications to discuss, and therefore students had an opportunity to discuss the similarities and difference between the different strategies. Investigating different approaches and evaluating solutions can create valuable learning opportunities in the mathematics classroom.

The Role of the Teacher

The learning opportunities discussed above were created, in large part, by the classroom teacher's actions. The classroom episodes discussed in this article require an environment that welcomes collaboration, open expression, and respect for ideas. Creating such a community requires a lot of support from the teachers. The teacher needs to facilitate classroom discussions by helping students clarify and articulate their ideas. The teacher needs to help the class as a whole assume the role as generators and evaluator of ideas. This includes asking students to share their thinking and helping them learn what type of information they need to offer to effectively share their reasoning with others. It also includes teaching students to take turns talking and listening, and to truly pay attention to one another's ideas. Reaching this level of discourse takes time and

practice, as well as modeling and support from the teacher. As students see such actions modeled, and practice engaging in them, students learn to share their thinking with one another and to listen to one another's ideas.

The fact that the teacher asked for multiple solution methods to the task was another important factor that set the stage for the learning opportunities that occurred in this class. However, the fact that the teacher listened to each student and respected their thinking helped the students produce multiple solutions. Had the teacher not respected each student's way of thinking, he might have pushed them into using the same type of reasoning and would not have received the diversity of solutions he received. Additionally, the fact that the teacher pushed the students to justify their strategies rather than just describe them played an important role in creating the learning opportunities afforded in this class. When students justify their answers, they share their thinking about a problem. This helps students realize that their thinking is valued in the classroom.

Listening to each other's justifications has other important implications as well. In particular, it helps students gain access to the thinking of others, which enables students to make sense of one another's reasoning (Kazemi & Stipek, 2001). Justifying the strategies meant that the students had opportunities to make sense of each other's thinking, which enabled them to discuss the merits of the different strategies. Had the students only shared their strategies without justifying them, students would have been less likely to discuss the different strategies offered and less likely to compare the different strategies (Wood, 2001). So asking the students to justify their answers set the

stage for the students to be able to discuss whether or not the proportional reasoning strategy worked.

Most important was the fact that the classroom teacher valued knowing why strategies work (or do not work in this case). This meant that when the students tried to argue that the proportional reasoning strategy was wrong simply because the students using it did not calculate the perimeter correctly, the teacher pushed for more. When he said to the students “that still doesn’t help us with what’s, what could be wrong with [this solution],” he pushed them to prove *why* the proportional reasoning method was wrong, and created an opportunity for learning. Had the students not been pushed to prove that the answer was wrong, they might not have had an opportunity to learn what was wrong with their answers or have had the opportunity to adjust their thinking to generate a correct solution strategy. Encouraging students to investigate mathematical statements even after they have found them to be true or false helps students learn about the mathematics in the problem, and generates a culture of sense making.

The Role of Proof in Mathematics

When discussing the role proof can play in the classroom, mathematics educators often look to the role proof plays in the field of mathematics (de Villiers, 1990). Maybe the most important role proof plays for mathematicians is that of *verification*. In the classroom vignette above, the initial purpose of the students’ proving is to verify the correctness of their answers. However, this resulted in opportunities for learning due to

the *communication* role of proving as the students attempted to prove and disprove the validity of the various strategies that were shared. The two different learning opportunities that are highlighted above also demonstrate the role of proving in providing explanation and in supporting discovery. Students had an opportunity to learn why the multiplicative strategy did not work because a student's proof that it did not work included an explanation of why. Further, as the student shared his argument he discovered a way to modify the proportional reasoning strategy and create a new viable strategy. In these ways, the role proof played in advancing understanding in this classroom is much like the role it plays in advancing understanding in the field of mathematics.

References

- Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 215-230). Dordrecht, NL: Kluwer.
- Aydeniz, M., Pabuccu, A., Cetin, P. S., & Kaya, E. (2012). Argumentation and students' conceptual understanding of properties and behaviors of gases. *International Journal of Mathematics Education Science and Technology*, 10(6), 1303-1324.
- Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 89-110). Dordrecht, NL: Kluwer.
- Balacheff, N. (2002). *The researcher epistemology: A deadlock for educational research on proof*. In F. L. Lin (Ed), *Proceedings of the International Conference on Mathematics: Understanding Proving and Proving to Understand*, Taipei, Taiwan.
- Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.
- Boaler, J. (2004). *Promoting equity in mathematics classrooms: Important practices and their impact on student learning*. Paper presented at the *International Conference on Mathematics Education*, Copenhagen, Denmark.

- Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed ability approach. *British Educational Research Journal*, 34(2), 167-194.
- Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers' College Record*, 110(3), 608-645.
- Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of Development in Geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.
- CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics* 1(1), 33-60.
- Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24(4), 359-387.
- Chazan, D., & Lueke, H. M. (2009). Exploring relationships between disciplinary knowledge and school mathematics: Implications for understanding the place of reasoning and proof in school mathematics. In D. Stylianou, M. Blanton & E. Knuth (Eds.), *Teaching and Learning Proof Across the Grades* (pp. 21-39). New York, NY: Routledge.
- Chin, E., & Tall, D. (2002). *Proof as a formal procept in advanced mathematical thinking*. In F. L. Lin (Ed), *Proceedings of the International Conference on Mathematics: Understanding Proving and Proving to Understand*, Taipei, Taiwan.

- Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28(3), 258-277.
- Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, 29(3), 573-604.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3/4), 175-190.
- Davis, P. (1986). *The nature of proof*. In M. Carss (Ed.), *Proceedings of the Fifth International Congress on Mathematical Education*, Birkhauser.
- de Villiers, M. (1986). The role of axiomatization in mathematics and mathematics teaching. In *RUMEUS Studies in Mathematics Education No. 2*: University of Stellenbosch.
- de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.
- Douek, N. (1999). Some remarks about argumentation and proof In I. Schwank (Ed.), *European Research in Mathematics Education I*. Osnabruck: Forschungsinstitut fur Mathematikdidaktik.
- Edwards, B., & Ward, M. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*, 111(5), 411-424.

- Edwards, L. (1999). Odds and evens: Mathematical reasoning and informal proofs among high school students. *Journal of Mathematical Behavior*, 17(4), 489-504.
- Elliot, R., Leissig, K., & Kazemi, E. (2009). Sociomathematical norms in professional development: Examining leaders' use of justification and its implications for practice. In L. Knott (Ed.), *The Role of Mathematics Discourse in Producing Leaders of Discourse*. Charlotte, NC: Information Age Publishing Inc.
- Fawcett, H. P. (1938). *The Nature of Proof*. New York, NY: National Council of Teachers of Mathematics.
- Fendel, D., Resek, D., Alper, L., & Fraser, S. (1997). *Interactive Mathematics Program*. Berkley, CA: Key Curriculum Press.
- Flores, A. (2002). How do children know that what they learn in mathematics is true. *Teaching Children Mathematics*, 8(5), 269-274.
- Forman, E. A. (2000). Knowledge building in discourse communities. *Human Development*, 43, 364-368.
- Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction*, 8(6), 527-548.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, HL: D. Reidel.
- Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140.
- Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21(1), 6-13.

- Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, 15(3), 42-49.
- Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1), 5-23.
- Hanna, G., & Barbreau, E. (2008). Proofs as bearers of mathematical knowledge. *ZDM*, 40, 345-353.
- Hanna, G., & Jahnke, H. N. (1993). Proof and application. *Educational Studies in Mathematics*, 24, 421-438.
- Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and Teaching Number Theory* (pp. 185-212). New Jersey: Ablex Publishing Corporation.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education*, 7, 234-283.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Learning and Teaching* (pp. 805-842). Charlotte, NC: Information Age Publishing.
- Herbst, P., & Balacheff, N. (2009). Proving and knowing in public: The nature of proof in a classroom. In D. Stylianou, M. Blanton & E. Knuth (Eds.), *Teaching and Learning Proof Across the Grades* (pp. 41-63). New York, NY: Routledge.

- Herbst, P., Miyakawa, T., & Chazan, D. (2012). Revisiting the functions of proof in mathematics classrooms: A view from a theory of instructional exchanges
Retrieved from <http://hdl.handle.net/2027.42/78168>.
- Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24, 389-399.
- Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65-100). New York: MacMillan Publishing Company.
- Initiative, C. C. S. S. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.
- Jacobs, J. K., Hiebert, J., Givvin, K. B., Hollingsworth, H., Garnier, H., & Wearne, D. (2006). Does eighth-grade mathematics teaching in the United States align with the NCTM Standards? Results from the TIMSS 1995 and 1999 video studies. *Journal for Research in Mathematics Education*, 37, 5-32.
- Jaffe, A. (1997). Proof and the evolution of mathematics. *Synthese*, 111(2), 133-146.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal*, 102(1), 59-80.
- Kidron, I., & Dreyfus, T. (2010). Justification enlightenment and combining constructions of knowledge. *Educational Studies in Mathematics*, 74, 75-93.
- Knuth, E. (2002a). Proof as a tool for learning mathematics. *Mathematics Teacher*, 95(7), 486-490.

- Knuth, E. (2002b). Secondary school mathematics teacher's conceptions of proof. *Journal for Research in Mathematics Education*, 33(5), 379-405.
- Knuth, E. (2002c). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61-88.
- Lai, Y., Weber, K., & Mejia-Ramos, J. P. (2012). Mathematicians' perspectives on features of a good pedagogical proof. *Cognition and Instruction*, 30, 146-149.
- Lakatos, I. (1976). *Proofs and Refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29-63.
- Lannin, J. (2005). Reasoning through patterning activities *Mathematical Thinking and Learning*, 3(3), 231-258.
- Lannin, J., Barker, D., & Townsend, B. (2006). Why, why should I justify. *Mathematics Teaching in the Middle School*, 11(9), 438-443.
- Lappan, G., Fey, J. T., Fitzgerald, W., Friel, S. N., & Phillips, E. D. (1998). *Connected Mathematics Series*. Palo Alto, CA: Dale Seymour.
- Larsen, S. (In press). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behavior*.
- Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, 67(3), 205-216.
- Leitao, S. (2000). The potential of argument in knowledge building. *Human Development*, 43(6), 332-360.

- Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Learning and Teaching* (pp. 763-804). Charlotte, NC: Information Age Publishing.
- Lucast, E. K. (2003). *Proof as method: A new case for proof in mathematics curricula*. Unpublished Masters Thesis, Carnegie Mellon University, Pittsburgh, PA.
- Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5 year case study. *Journal for Research in Mathematics Education*, 27(2), 194-214.
- Mamona-Downs, J., & Downs, M. (2005). The identity of problem solving. *Journal of Mathematical Behavior*, 24, 385-401.
- Martin, D. B. (2003). Hidden assumptions and unaddressed questions in mathematics for all rhetoric. *The Mathematics Educator*, 13(2), 7-21.
- Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, 15, 277-289.
- Mejia-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. d. Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88-93). Taipei, Taiwan.
- Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27(3), 249-266.
- National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: National Council of Teachers of Mathematics.

- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Ogan-Bekiroglu, F., & Eskin, H. (2012). Examination of the relationship between engagement in scientific argumentation and conceptual knowledge. *International Journal of Science and Mathematics Education, 10*(6), 1415-1443.
- Otte, M. (1994). Mathematical knowledge and the problem of proof. *Educational Studies in Mathematics, 26*(4), 299-321.
- Piaget, J. (1971). *Genetic Epistemology* (E. Duckworth, Trans.). New York, NY: W. W. Norton & Company.
- Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof. *Educational Studies in Mathematics, 52*, 319-325.
- Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education, 37*, 388-420.
- Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7*, 51-73.
- Reid, D. A. (1995). Proving to explain. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th Conference of the International Group of Psychology of Mathematics Education* (Vol. 3, pp. 137-143). Recife, BR.
- Reid, D. A. (2011). Understanding Proof and Transforming Teaching In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American*

Chapter of the International Group for the Psychology of Mathematics Education.
Reno, NV.

- Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 334-370). New York, NY: MacMillan Publishing.
- Schroeder, T. L., & Lester, F. K. (1989). Developing understanding in mathematics via problem solving. In P. R. Trafton (Ed.), *New Directions for Elementary School Mathematics, 1989 Yearbook of the National Council of Teachers of Mathematics* (pp. 31-42). Reston, VA: NCTM.
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem. *Journal for Research in Mathematics Education*, 34(1), 4-36.
- Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27(2), 4-13.
- Sierpinska, A. (1994). *Understanding in Mathematics*. London, UK: Falmer Press.
- Simon, M., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, 15(1), 3-31.
- Skemp, R. (1971). *The Psychology of Learning Mathematics*. Harmondsworth, UK: Penguin Books.
- Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior*, 24, 341-350.

- Staples, M., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifaceted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, 31(4), 447-462.
- Staples, M., & Truxaw, M. (2009). *A journey with justification: Issues arising from the implementation and evaluation of the Math ACCESS project*. In Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.), *Proceedings of the thirty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 5 (pp. 827-835). Atlanta, GA: Georgia State University.
- Steen, L. (1999). Twenty questions about mathematical reasoning. In L. Stiff (Ed.), *Developing Mathematical Reasoning in Grades K-12: 1999 Yearbook* (pp. 270-285). Reston, VA: National Council of Teachers of Mathematics.
- Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior*, 21, 459-490.
- Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (Second ed.). Thousand Oaks, CA: Sage Publications.
- Stylianides, G. J. (2005). *Investigating students opportunities to develop proficiency in reasoning and proving: A curricular perspective*. Unpublished Doctoral Dissertation, The University of Michigan, Ann Arbor, MI.
- Stylianides, G. J., & Stylianides, A. J. (2006). "Making proof central to pre-high school mathematics is an appropriate instructional goal": Provable, refutable, or undecidable proposition. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková

- (Eds.), *Proceedings of the 30th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 209-216). Prague, Czech Republic.
- Stylianou, D., Blanton, M., & Knuth, E. (Eds.). (2009). *Teaching and Learning Proof Across the Grades: A K-16 Perspective*. New York, NY: Routledge.
- Tall, D. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-511). New York, NY: MacMillan Publishing.
- TERC. (1998). *Investigations in number, data, and space*. Palo Alto, CA: Dale Seymour Publications.
- Thanheiser, E., Staples, M., Bartlo, J., Heim, K., & Sitomer, A. (2010). *Justification in middle school classrooms: How do middle school teachers define justification and its role in the classroom?* In Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.), *Proceedings of the thirty-second annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 6 (pp. 860-868). Columbus, OH: The Ohio State University.
- Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics*, 15(1), 29-37.
- Toulmin, S. (1969). *The uses of argument*. Cambridge: Cambridge University Press.

- Uhlig, F. (2002). The role of proof in comprehending and teaching elementary linear algebra. *Educational Studies in Mathematics*, 50, 335-346.
- Vanderhye, C., & Zmijewski, C. (2007). Assessing student understanding through conversations. *Teaching Children Mathematics*, 14(5), 260-264.
- von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Learning* (Vol. 6). London: RoutledgeFalmer.
- Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior*, 24, 351-360.
- Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, 39(4), 431-459.
- Weber, K. (2010). Proofs that develop insight. *For the Learning of Mathematics*, 30(1), 32-36.
- Weber, K., Maher, C., Powell, A., & Lee, H. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, 68, 247-261.
- Whitenack, J., & Knipping, N. (2002). Argumentation, instructional design theory and students' mathematical learning: A case for coordinating interpretive lenses. *Journal of Mathematical Behavior*, 21, 441-457.
- Whitenack, J., & Yackel, E. (2002). Making mathematical arguments in the primary grades: The importance of explaining and justifying ideas. *Teaching Children Mathematics*, 8(9), 524-527.

- Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math?: Resources and acts for constructing and understanding mathematics *Journal of Mathematical Behavior*, 78, 21-43.
- Wood, T. (1996). Events in learning mathematics: Insights from research in classrooms. *Educational Studies in Mathematics*, 30(1), 85-105.
- Wood, T. (1999). Creating a context for argument in a mathematics class. *Journal for Research in Mathematics Education*, 30(2), 171-191.
- Wood, T. (2001). Teaching differently: Creating opportunities for learning mathematics. *Theory into Practice*, 40(2), 110-117.
- Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, 37(3), 222-255.
- Wu, H. (1996). The role of Euclidean geometry in high school. *Journal of Mathematical Behavior*, 15, 221-237.
- Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior*, 21, 423-440.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.
- Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 227-236). Reston, VA: National Council of Teachers of Mathematics.

Yin, R. K. (2006). Case study methods. In J. L. Green, G. Camilli & P. B. Elmore (Eds.),
Handbook of Complementary Methods in Education Research (pp. 111-122).

Mahwah, NJ: Erlbaum.

Yopp, D. (2011). How some research mathematicians and statisticians use proof in
undergraduate mathematics *Journal of Mathematical Behavior*, 31, 115-130.